Hodge Theory in Grassmannians

by

Enrico Fatighenti

Submitted to The University of Warwick

for the degree of

Doctor of Philosophy

Warwick Mathematics Institute

November 2017

THE UNIVERSITY OF WARWICK

Contents

Acknowledgments

I wish to thank my advisor Miles Reid for teaching me how to become a mathematician. One of the most important lesson I learned from him is that research is not about watching a football game, but rather kicking the ball and start playing. My work (and my taste in mathematics as junior researcher) has been hugely shaped by his ideas and his view of mathematics. Last but not least, I would like to thank him for his energy in organising events at Warwick: I have definitely learned more from random conversations with established mathematicians than from attending formal courses.

A special thanks to Domenico Fiorenza for his friendship, his constant advice and support, and for believing in my very first idea. I would like to thank Edoardo Sernesi for his enthusiasm and willingness to answer every possible stupid question I posed to him. Many other friends and collaborators had a great impact on my work and helped my in various stages during the last four years. I would like to mention in particular Hamid Ahmadinezhad, Christian Böhning, Gavin Brown, Alessio Corti, Carmelo Di Natale, Tim Logvinenko, Luca Migliorini, Giovanni Mongardi and Francesco Zucconi.

Many thanks to all the other PhD in Algebraic Geometry that I have met in Warwick in this four years. Let me mention in particular Sara, Eduardo, Mike, Seung-Jo and Tom. Each one of these people had a great impact on me, both from a mathematical and human point of view. I would like to thank as well all my other friends and colleagues from Warwick. A special thank to John Cremona, for allowing me to use the Number Theory server for my often obscenely long calculations.

During my PhD I have been a guest of KIAS and University of Nairobi. I would like to thank all the people at both institutions (and surroundings) that spent some of their time in making my life easier. I would like to mention in particular Yoonsuk Hyun and Jared Ongaro. I look forward to come back to both places soon.

* * *

La seconda parte dei ringraziamenti è sicuramente la più sentita. In italiano, come le persone a cui è dedicata. Un grazie infinito a tutti voi del Pollo (questa volta inteso come gruppo, e non come luogo). Se non si fosse creata questa atmosfera fantastica il dottorato sarebbe stata una esperienza molto meno sopportabile. Ognuno di voi meriterebbe un ringraziamento speciale, e comunque non renderei giustizia. Spero solo che una piccola enclave di Garbatella possa rimanere ancora a lungo a Leamington Spa. Ringrazio anche tutti i membri onorari del nostro gruppo, come Aurelio e Damiano, e tutti coloro che sono passati da qui negli ultimi anni. Ringrazio anche Siringia, i the Grevis e i Pelegrinos, persone che c'erano da prima e continueranno a rimanere.

Voglio ringraziare tutte le persone che ho conosciuto in questi anni di conferenze e viaggi. In particolare ringrazio (tutti i) Levico Terme e i GAeL a cui ho partecipato.

Qualche menzione speciale: Claudio e Roberto 'Bob' Theface, che hanno organizzato con me pressoché ogni cosa negli ultimi anni. Camilla, per l'ospitalità meravigliosa a Bologna. Andrea e Riccardo, per aver sopportato me e Bob in Norvegia, Vlancu, per esser sceso dal camion e imbracciato l'ascia.

Infine, le dediche più importanti. Ad Antonella e Fausto, per essere stati i migliori genitori anche a distanza. E a Serena, perché senza di lei, probabilmente, questa tesi non sarebbe mai nata.

Declarations

I declare that the work contained in this thesis is original except where otherwise stated in the text. I confirm that this thesis has not been submitted anywhere else for any degree. The results of chapter 2 are adapted from the preprint [59], joint with with Luca Rizzi and Francesco Zucconi. Chapter 3 is adapted from the preprint [52], joint with Carmelo Di Natale and Domenico Fiorenza. Chapter 4 is adapted from [22], joint with Gavin Brown. All the aforementioned preprints are available on the arXiv.

Abstract

In this thesis we study several generalisations of the *Griffiths's residue technique.* We first show how the deformation modules T^i of the affine cone over a smooth projective variety X contain the Hodge groups of X as homogeneous slices. We discuss several applications, mainly in the Birational Geometry of $\mathbb{Q}$-Fano threefolds. We then investigate the case of subvarieties of the Grassmannian $\mathrm{Gr}(k, n)$. For an hypersurfaces (or a complete intersection) X in the Grassmannian $\mathrm{Gr}(k, n)$ we are able to expliciteIy construct a *Griffiths ring* that allows us to compute all the Hodge groups $H^{p,q}(X)$. We then apply our techniques to construct new interesting varieties in the Grassmannians, such as surfaces of general type with low invariants and Fano manifolds of K3-type.

Introduction

Outline of the thesis

Pre-history of the problem

Hodge theory is a powerful tool for solving problems in geometry. Its original aim is to study the cohomology of a smooth compact manifold by the means of a system of PDE and differential forms. Despite its analytic origin Hodge theory has rapidly fallen into the realm of algebraic geometry, thanks to the work of Deligne, Griffiths and many others. In particular many of the original analytic and differential tools can be transformed in more algebro-geometric-friendly objects, such as coherent cohomology groups $H^p(\Omega^q_X)$. The natural set-up for Hodge theory is the category of smooth compact Kähler manifold: thanks to Chow's theorem all smooth projective variety are actually examples. In this thesis, we do not care about non-algebraic objects: in particular we only consider projective objects and the words *differential forms* will appear very rarely. An introduction of the basics of Hodge theory can be found in 1.1.

Torelli problem and Griffiths calculus. Torelli-type problems are amongst the most natural questions arising in Hodge theory. One starts by associating to a smooth projective variety X its Hodge structure. This is given in turn by a bunch of linear algebra and lattice theory data. Very roughly speaking, a Torelli theorem determines (to what extent) these data are actually enough to completely characterise X itself. The history of Torelli-type theorems is rich both in examples, counterexamples and partial results. We discuss the question in detail in 1.3. One of the main ingredient needed for Torelli-type results is the explicit determination of the Hodge groups. In fact just knowing the dimension of the complex vector spaces $H^q(\Omega^p_X)$ is rarely enough. One needs often to have explicit (for example polynomial) generators, in order to understand better the whole ring structure. In this context one of the key results is Griffiths's computation of the Hodge filtration of an hypersurface in terms of the the cohomology groups of the ambient space. Roughly speaking, if X is a smooth projective hypersurface, Griffiths's result associates to X a certain graded ring R (the so-called *Jacobian ring*) and identifies some homogeneous slices of R with the Hodge groups of X. This is an enormous advantage for many Hodge theoretical calculations, and is indeed crucial in the proof of Torelli theorems for hypersurfaces, [55] and many other results. Griffiths's result is described in 1.2. Over the course of the past forty years some generalisations have been achieved, for example smooth complete intersections in projective space and hypersurfaces in toric varieties, see [53], [9]. However, a general picture is still missing. We can therefore state (only loosely at this stage) the main motivation for this thesis.

Problem 1. Let $X \subset P$ be a smooth subvariety of a nice projective variety P. What properties do we have to ask for X, P in order to have an algebraic object A controlling (part of) the Hodge theory of X?

An example of "nice" could be smooth and endowed with suitable cohomological vanishings. We could impose some algebraic properties for X, for example its coordinate ring being complete intersection or Gorenstein with respect to the one of P. Again, the algebraic object A could be a graded ring in the best option, or maybe a module. In this thesis we propose some precise settings for the problem above. Before giving a recap of the results, I want to present some further motivation for this work. Namely, which applications lie within the range of our results?

Some applications of Hodge theory (and more motivations for this thesis)

Hodge theory and birational geometry: classification of Fano threefolds. One of the main tools in birational geometry is the Minimal Model Program (MMP). The ultimate goal of the MMP is to generalise as much of the Enriques–Kodaira classification of algebraic surfaces to higher dimensions as possible. Unlike the case of surfaces, the elementary operations associated with the MMP algorithm often bring the result outside the category of smooth varieties. Therefore we consider varieties with terminal singularities— the smallest class of singularities that are preserved under these operations. Although these varieties are singular from a classical viewpoint, the usual game is to consider them as if they were smooth. For example, they carry a natural pure Hodge structure, see [54] and 4.0.1. The case of Fano 3-folds is one of the most relevant in this setting. Thanks to [76] the classification of Fano 3-folds consists of finitely many deformation families. However, they are far from being classified. In the online Graded Ring Database [24] almost 50.000 numerical candidates are collected. In low codimension actual families of Fano 3-folds are constructed starting from these Hilbert series, and the list is exhaustive. A fundamental question is either prove or disprove the existence of these candidates, and a complete understanding of their Hodge theory could provide useful insights. In chapter 4 we carry out the first steps of this programme.

Link with hyperkähler geometry, derived categories, Fano manifold of Calabi-Yau type and so on. The hunt for new examples of hyperkähler varieties motivates the study of Fano varieties whose middle Hodge structure contains a sub-Hodge structure of weight 2. The ancestral (and main) example is the cubic fourfold $X_3 \subset \mathbb{P}^5$, whose $H^4(X_3, \mathbb{C})$ admits a decomposition as $\mathbb{C} \oplus \mathbb{C}^{21} \oplus \mathbb{C}$, thus resembling the Hodge diamond of a K3 surface. To a cubic fourfold we associate its Fano varieties of lines $F_1(X_3)$, that can be seen as the zero set of a generic section of the bundle $\mathrm{Sym}^3 \mathcal{S}^*$ over Gr(2,6). A famous result of Beauville-Donagi shows that $F_1(X_3)$ is an hyperkähler variety. More in general, a result of Kuznetsov and Markushevich, [81] shows that to any complex projective variety X of dimension n and $\mathfrak{M}$ a moduli space of stable or simple sheaves on X, then any form in $H^{n-q-2}(X, \Omega^{n-q})$ defines a closed 2-form in $H^0(\mathfrak{M}^{\text{smooth}}, \Omega^2)$. This explains the search for varieties with cohomological properties similar to the cubic fourfold. Few examples are known, as the Gushel-Mukai fourfold $X_{2,1} \subset \mathrm{Gr}(2,6)$, [47], [48], [49] and the linear twenty-fold hypersurface $Y_1 \subset \mathrm{Gr}(3,10)$ of Debarre-Voisin, [50]. Küchle classification [79] of Fano fourfolds of index 1 as section of homogeneous vector bundles on Grassmannians is also relevant, since it contains a few more examples. In a series of very recent works Kuznetsov has made as well explicit the link between the geometry of these Fanos and Derived categories, primarily in connection with rationality questions ([82], [49]). A generalisation of these idea is the concept of *Fano varieties of Calabi-Yau type*. These are particular types of Fano, introduced by Iliev and Manivel in [69], whose middle Hodge structure contains a weight k sub-Hodge structure of CY-type. A reasonably large class of new examples (if not all of them) of these varieties should come from homogeneous vector bundles over homogeneous spaces. A Griffiths-type result would transform the problem into a purely algebraic one, therefore simplifying the hunt. In chapter 8 and in the last chapter we deal with this problem.

Explicit construction of Calabi-Yaus and Surfaces of general type. The close study of Fano varieties in high dimension is a good starting point for the classification of Calabi-Yau threefolds and surfaces of general type. Indeed with an appropriate choice of dimension and index a Fano X can act as a *key variety* for a CY 3-fold and a surface of general type. The classification of the first one is one of the longstanding problems in algebraic geometry. It is still not know if the number of deformation families of CY3 fold is finite or not, and even simple geometrical properties of the moduli spaces are not known (cf. Reid's fantasy, [98]). For surfaces of general type the situation is even murkier: many examples are known, especially with small invariants p_g and q. However a systematic classification is missing. On the other hand CY 3-folds and surfaces constructed from the above Fano enjoys many symmetries, coming from the ambient homogeneous spaces. The idea is therefore to use some straightforward geometric constructions to produce new examples out of old ones. In chapter 6 we present some new constructions and the general manifesto.

Results in this Thesis, part I

Infinitesimal Torelli problem for $\mathbb{Q}$-factorial, terminal Fano threefold hypersurfaces. The first interplay between problems in birational geometry and Hodge theory that we present is an infinitesimal version of the Torelli theorem for Fano 3-fold hypersurfaces with $\mathbb{Q}$-factorial terminal singularities. This is a nice warm-up and serves as a playtest of many techniques used in this thesis. A classical version of the infinitesimal Torelli theorem for quasi-smooth weighted hypersurface was partially given by Tu in [115]. The 135 families of $\mathbb{Q}$-Fano 3-fold hypersurfaces are amongst the cases not covered. Through a detailed case-by-case analysis we check that all the "famous 95" $\mathbb{Q}$-Fano threefold of index 1 satisfies the (generic) infinitesimal Torelli theorem. The situation becomes much more interesting in the index > 1 case. This is indeed summarised by the following theorem.

Theorem 1 (Theorem 2.1.11)**.** Let $\mathcal{M}$ be the space of quasi-smooth weighted hypersurfaces of degree d in $\mathbb{P}$ modulo automorphisms of $\mathbb{P}$, for any of the 28 families of quasi-smooth Fano Threefolds of index $i_X > 1$ with non trivial Hodge structure. Then

- for the families no. 115, 121, 122 and 127 the infinitesimal Torelli theorem does not hold;
- for the remaining 24 families, there is an open dense subset of $\mathcal{M}$ on which the infinitesimal Torelli theorem holds.

We show as well that when the dimension increases, the situation only gets worse. We introduce in fact a straightforward geometrical construction, based on the *double suspension trick*. Indeed starting from any of the above Fano threefolds $X = X^1$, we can create an infinite chain of double covers where any new Fano X^{k-1} sits as ramification divisor for the next double cover $X^k \stackrel{2:1}{\to} w\mathbb{P}$. This introduce a periodic pattern in the chain, with an isomorphism of IVHS between even and odd member of the chain. We perform the following construction.

Theorem 2 (Theorem 2.2.7)**.** A general quasi-smooth Fano hypersurface $X_{14}^{2k+1} \subset \mathbb{P}(2,3,4,5,7^{2k+1})$ will be a counterexample for the Torelli theorem. On the other hand the middle Hodge structure of $X_{14}^{2k} \subset \mathbb{P}(2,3,4,5,7^{2k})$ will be of K3-type. In particular the infinitesimal Torelli theorem holds.

We have then an infinite chain of examples and counterexamples for the Torelli problem, with alternate dimensions. This is another confirmation of the pathological behaviour of Fano varieties, and rules out the possibility of a general answer for the Torelli problem even in simple cases.

Hodge theory and deformations of affine cones. In chapter 3 we present the first important generalisation of problem 1. We make very few assumptions on the algebraic properties of X, namely a subcanonicality-type condition. We do not ask for a specific codimension as well. The object that we consider is the graded module $T^1_{A_X}$, that controls the infinitesimal, first order deformations of the affine cone A_X over X. This module is a direct generalisation of the Jacobian ring for an hypersurface. The main result is the following.

Theorem 3 (Theorem 3.1.4)**.** Let X be a smooth subcanonical projectively normal variety of dimension $n > 1$, and let $m \in \mathbb{Z}$ be the integer such that $\omega_X \cong \mathcal{O}_X(m)$. If $H^1(X, \mathcal{O}_X(k)) = 0$ for every $k \in \mathbb{Z}$, then we have

$$(T^1_{A_X})_m \cong H^{n-1,1}_{\text{prim}}(X)$$

Equivalent statements can be given for the other graded components of the module. This answer is particularly satisfying for computational purposes, but is far from being complete. Indeed, we would like to have full ring (and not just a module) containing all the other Hodge groups. A first solution can be borrowed from the realm of derived categories.

Theorem 4 (Theorem 3.2.3)**.** Let X as above, U_X the punctured affine cone over X, and $\mathrm{HH}^\bullet(U_X)$ denoting the Hochschild cohomology of the derived category of coherent sheaves over U_X. Then

$$\mathrm{HH}^{p,q}(U_X)_m \cong H^{n-p+1,q}_{\text{prim}}(X) \oplus H^{n-q,p}_{\text{prim}}(X).$$

Hodge theory and birational geometry: the Fano threefold case. The results of chapter 3 have several nice and practical applications. The first one is the computation of Hodge numbers of $\mathbb{Q}$-Fano threefolds in low codimension. We compute all the Hodge numbers up to codimension 3, and a bunch of significative examples in codimension 4. The latter are particularly significative: indeed we obtain for the first time a confirmation of the existence of families with Picard rank $\rho > 1$. We use a mixture of birational tools, T^1-technique and a nice relation between the Hodge theory and the deformation theory of a Fano threefold, namely

Theorem 5 (Theorem 4.1.2)**.** Let X be a Fano 3-fold with K3 elephant $E \subset X$ and genus $g_X = h^0(X, -K_X) - 2$.

1. Setting $\alpha_E = h^{1,1}(E) - g_X + 1$,

$$h^1(X, T_X) - h^0(X, T_X) = \alpha_E + h^{2,1}(X) - h^{2,2}(X). \tag{1}$$

2. If X is a complete intersection in weighted projective space or in a weighted Grassmannian $w\,\mathrm{Gr}(2,5)$, then $h^0(X, T_X) = 0$.

Results in this thesis, part II

We move then to the detailed study of homogeneous spaces. Although our methods are built to be fairly general, we mostly focus on the Grassmannian case.

Varieties of low dimension in Grassmannians. The classification of subvarieties in Grassmannian as zero locus of a section of an homogeneous vector bundle is a widely open problem in algebraic geometry. Partial results have been obtained in the case of Fano fourfold, and varities with trivial canonical bundle in dimension 3 and 4, see [79], [71], [13]. The idea is that many of the members in the previous classifications could function as key varieties for new Calabi-Yau threefolds and surfaces of general type. In fact many of these families come equipped with several symmetries, and suitably good group actions. As an example, we first construct

Theorem 6 (Theorem 6.1.8)**.** 1. Let W a seven-codimensional linear section of the Grassmannian Gr(2,7) constructed by taking an invariant $\mathbb{P}^{13}$ as in 6.1.1. Then W admits a free $\mathbb{Z}/7$ action. In particular the quotient $\pi : W \to \widetilde{W}$ yields a smooth Calabi-Yau threefold;

2. Let $\widetilde{S}$ the surface of general type obtained by intersecting $\widetilde{W}$ with a further $\mathbb{Z}/7$-invariant hyperplane section. Then $p_g(\widetilde{S}) = 1$, $q(\widetilde{S}) = 0$, $K^2_{\widetilde{S}} = 6$.

The next step of the construction would be to find a fixed-point free involution on the surface $\widetilde{S}$, and therefore construct a surface with $p_g = q = 0, K^2 = 3$. We plan to deal with this case, and all the others listed in 6.1 in the near future.

A Griffiths residue-type theorem for hypersurfaces in homogeneous spaces. In this chapter we provide a further answer to the main problem (1). The main result here is the explicit determination of an equivalent of the Jacobian ring for hypersurfaces in Grassmannian $\mathrm{Gr}(k, n)$ as quotient of a polynomial ring. Suppose that an hypersurface X is given by the zero set of an homogeneous polynomial f of degree d in the Plücker ring. Let the Lie algebra $\mathfrak{sl}_n$ acts on the coordinate ring of the Grassmannian $\mathrm{Gr}(k, n)$ as derivations. We define our *Griffiths ring* R^G_f as a quotient of the Plücker algebra S by the ideal generated by those derivations, see 7.1. Let $I_{j-1,j}$ as the cokernel of the map

$$0 \to H^{j-1,j-1}(\mathrm{Gr}(k, n)) \to H^{j,j}(\mathrm{Gr}(k, n)).$$

Theorem 7 (Theorem 7.1.8)**.** Let $X_d = V(f)$ a smooth hypersurface in the Grassmannian $G = \mathrm{Gr}(k, n)$. Let $N = \dim(G) = k(n-k)$, and R^G_f the Jacobian ring for X defined in 7.1.2. Assume that $d \geq n - 1$. If $\dim(X) = N - 1 \equiv 0 \ (2)$. Then

$$[R^G_f]_{(p+1)d-n} \cong H^p_{\mathrm{prim}}(X, \Omega^{N-1-p}).$$

If $\dim(X) = N - 1 \equiv 1 \ (2)$ then

$$[R^G_f]_{(p+1)d-n} \cong H^p_{\mathrm{prim}}(X, \Omega^{N-1-p}) \oplus \delta_{p,\frac{N}{2}} I_{p-1,p},$$

where $\delta_{p,\frac{N}{2}}$ is the Kronecker delta symbol.

When the positivity condition above is not verified, some residual cohomological contribution from the Grassmannian might appear. We give weaker version of the result even in this case, especially in the case of the Grassmannian of lines. Further conditions might be deduced by a careful analysis of the Borel-Bott-Weil algorithm.

Extension to the complete intersection case. In chapter 8 we extend the result of the above section to the case of smooth complete intersections in Grassmannian. The idea is to use the standard *Cayley trick*, as in the projective case. We define a suitable version of the Griffiths ring $\mathcal{U}$ for complete intersections in 8.4. The result is

Theorem 8 (Theorem 8.1.4)**.** Let $Z = Z_{d_1,\dots,d_c}$ a smooth complete intersection in a Grassmannian Gr(k,n), and let $\mathcal{U}$ the Griffiths ring attached to Z. Denote by m the canonical degree of Z, that is $\omega_Z \cong \mathcal{O}_Z(m)$. Suppose $m \geq n-1$. Then if $\dim(Z) = N-c$ is even

$$\mathcal{U}_{p,m} \cong H^{N-c-p,p}_{\text{prim}}(Z).$$

If $\dim(Z) = N-c$ is odd

$$\mathcal{U}_{p,m} \cong H^{N-c-p,p}_{\text{prim}}(Z) \oplus \delta_{p,\frac{N-c}{2}} I_{p,p-1}(G).$$

A bunch of significative examples is computed, as for example the Gushel-Mukai fourfold $X_{2,1} \subset \mathrm{Gr}(2,5)$.
We end this thesis with a chapter collecting a list of partial results and potential applications of the above theorem, all of them linked with the already listed motivations in this introduction. In particular we collect few more examples of Fano varieties of high dimension and index whose middle Hodge-structure is of K3-type. As an example we have the following result.

Proposition 1 (Proposition 9.0.4)**.**

1. Let $Z_1 \subset \mathrm{OGr}(3,8)$ given by a generic linear section. Then $h^{p,q} = 0$ for $p \neq q$ and $p+q < 8$. For $p+q = 8$, we have

$$h^{5,3} = h^{3,5} = 1,\ \ h^{4,4} = 24,$$

with all the others middle Hodge numbers equal to zero. Moreover $h^{4,4}_{\text{prim}}(Z_1) = 19$;

2. Let $W_1 \subset \mathrm{SGr}(3,9)$ given by a generic linear section. Then $h^{p,q} = 0$ for $p \neq q$ and $p+q < 14$. For $p+q = 14$, we have

$$h^{6,8} = h^{8,6} = 1,\ \ h^{7,7} = 26,$$

with all the others middle Hodge numbers equal to zero. Moreover $h^{7,7}_{\text{prim}}(Z_1) = 20$.

Notation

We always work over $\mathbb{C}$, the field of complex numbers. All varieties are assumed to be smooth and projective unless stated otherwise. We introduce some notations that we will use throughout this thesis.

- $\mathrm{Gr}(k,n)$ (sometimes shortened with G in long expressions) will always denote the

Grassmannian of k dimensional subspaces in a vector space $\mathbb{C}^n$. It will be considered as a projective variety of dimension $N = k(n-k)$ under the Plücker embedding, with $\mathcal{O}_G(1)$ defined accordingly. We denote with $\mathcal{S}$ and $\mathcal{Q}$ the rank k tautological and the rank $n-k$ quotient bundle over Gr(k,n).

- $X_d \subset \mathbb{P}(a_0, \ldots, a_n)$ (or $Z_d,\ S_d, \ldots$) will denote a generic hypersurface of degree d of the weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$. If the weights are non necessary, we may shorten the notation for the ambient projective space and write simply $w\mathbb{P}$. Similarly, $X_{d_1,\ldots,d_c}$ will denote a generic complete intersection of multi-degree $(d_1, \ldots, d_c)$.

- We will use in an interchangeable way the notation K_X and ω_X for the canonical divisor (resp. the canonical sheaf).

- Unless stated otherwise, we will use the letter R for the Jacobian (or Griffiths) ring associated to a smooth variety X. The type and definition of Griffiths ring will depend by the context, and will be specified each time. If $X = V(f)$ is an hypersurface, its Jacobian ring will be denoted R_f.

- If X is a smooth projective variety, A_X will denote the affine cone over X.

- The tangent bundle of X will be denoted by T_X. On the other hand $T^1_{A_X}$ will denote the T^1 module of the affine cone A_X. The subscript $(T^1_{A_X})_k$ will refer to the $k-$th graded component of the module.

- We will only display half of the Hodge diamond of a variety. This is motivated by the frequent tours in (very) high dimensional algebraic geometry in this thesis.

- Codes and various computer algebra material used in this thesis can be accessed via [?].

Part I

The projective and weighted projective worlds

Chapter 1

Preliminaries

In this section, we review most of the useful results that we will use throughout the first (and sometimes second) part of this thesis. In particular we will focus on the definition of Hodge structure and on some Torelli-type theorems.

1.1 Introduction to Hodge Theory

Hodge theory is one of the main tool of complex algebraic geometry. It provides a way to associate to a smooth projective variety a distinguished set of invariants, somehow easier to calculate. As we will see in few pages, sometimes this is enough to reconstruct the variety itself. Let us start with a formal definition.

Definition 1.1.1. An Hodge structure of weight k is given by a finitely generated abelian group $H_{\mathbb{Z}}$ such that its complexification $H := H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ carries a decomposition

$$\begin{cases} H^{p,q} = \overline{H^{q,p}} \\ H = \bigoplus_{p+q=k} H^{p,q} \end{cases}$$

Associated to an Hodge structure there is a canonical *Hodge filtration*

$$H^k = F^0 H^k \supset F^1 H^k \supset \ldots \supset F^k H^k \supset \{0\},$$

defined by $F^p H = \bigoplus_{r \geq p} H^{r,k-r}$ and from this definition it follows that we have

$$Gr^p F^\bullet H := \frac{F^p}{F^{p+1}} = H^{p,k-p}.$$

The main (in our context, unique) example for an Hodge structure are the cohomology groups (for every k) with integer coefficient $H^k(X,\mathbb{Z})$ of a smooth projective variety X, once we eventually mod out by any torsion. Actually, projectivity is not a strictly necessary assumption: everything works as well in the more general context of compact Kähler manifold. However we do not care about non-algebraic objects. The Dolbeault cohomology spaces, for any complex manifold X can be defined as

$$H^{p,q}(X) := \frac{\ker \bar{\partial} : \Gamma(\Omega^{p,q}(X)) \to \Gamma(\Omega^{p,q+1}(X))}{\mathrm{Im}\,\bar{\partial} : \Gamma(\Omega^{p,q-1}(X)) \to \Gamma(\Omega^{p,q}(X))}$$

$\Gamma(\Omega^{p,q}(X))$ being (p,q)-forms on X (more precisely, the sections of the sheaf of complex differential forms of degree p,q), that is of type $\sum_{|I|=p,|J|=q} \alpha_{IJ} dz_I d\bar{z}_J$. Thanks to Dolbeault's isomorphism, we can rewrite the previous space in a more algebro-geometric friendly way

$$H^{p,q}(X) \simeq H^q(X, \Omega^p_X).$$

As usual we will collect all the information about the dimension of cohomology group ($h^{p,q}$:=dim$H^{p,q}$) of our manifold in the so-called *Hodge diamond*

$$\begin{array}{ccccccccc}
 & & & & h^{n,n} & & & & \\
 & & & h^{n-1,1} & & h^{1,n-1} & & & \\
 & & \cdots & & \cdots & & \cdots & & \\
h^{n,0} & & & \cdots & & \cdots & & & h^{0,n} \\
 & & \cdots & & \cdots & & \cdots & & \\
 & & & h^{0,1} & & h^{1,0} & & & \\
 & & & & h^{0,0} & & & &
\end{array}$$

with the obvious symmetries $h^{p,q} = h^{q,p} = h^{n-p,n-q} = h^{n-q,n-p}$. One of the main useful tool to compute these numbers is the famous *Lefschetz hyperplane section theorem* : we don't state here the complete result, that can be found in any classical book of algebraic geometry, but just a corollary that is relevant here.

Lemma 1.1.2 (Corollary of Lefschetz's theorem)**.** *Let $X \subset P$ a smooth complete intersection in a smooth variety P. Then $H^{p,q}(X) \cong H^{p,q}(P)$ for any $p+q < \dim X$.*

As an example, consider a smooth hypersurface in $\mathbb{P}^{n+1}$. Since it is well known that

$$h^{p,q}(\mathbb{P}^{n+1}) \begin{cases} 1 & \text{if } p = q \\ 0 & \text{otherwise} \end{cases}$$

we obtain that the Hodge diamond of a smooth projective hypersurface is

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & \dots & & \dots & & \dots & & \dots & \\
h^{n,0} & & \dots & & \dots & & \dots & & h^{0,n} \\
 & \dots & & \dots & & \dots & & \dots & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

that is, the only "interesting" line is the central one. Before doing the actual work, we need one more definition. Let ω be a Kähler class associated to a hypersurface X of dimension n: we define the *primitive cohomology* as

$$H^n_{\text{prim}}(X) := \operatorname{Ker} \lambda : (H^n(X) \to H^{n+2}(X))$$

where $\lambda(\alpha) = \omega \wedge \alpha$ and is an operator of type (1,1). Since the most natural choice for a Kähler class of a projective variety is simply the restriction of $[H]$ to X, the hyperplane class $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$ induced by the Fubini-Study metric, we can think of the primitive cycless as the one that do not intersect $[H]|_X$. Primitive cohomology will play a major role in the next pages; the key point is that its knowledge, for an hypersurface, actually implies the knowledge of the ordinary cohomology. In particular, just using the definition, we have immediately that, if the dimension n of X is odd, then $H^n_{\text{prim}}(X) = H^n(X)$, while if n is even we have

$$\begin{cases} H^{p,q}_{\text{prim}}(X) = H^{p,q}(X) & \text{if } (p,q) \neq (\frac{n}{2}, \frac{n}{2}) \\ H^{p,q}_{\text{prim}}(X) \oplus \mathbb{C} = H^{p,q}(X) & \text{otherwise} \end{cases}$$

Another interesting subspace of $H^n(X)$ is the *vanishing* cohomology. If X is embedded in a smooth projective variety P as before, and l_* is the (induced) Gysin morphism in cohomology we define

$$H^n_{\text{van}}(X) := \operatorname{Ker}(l_* : H^n(X) \hookrightarrow H^{n+2}(P)).$$

If X is an ample hypersurface in P and P has no primitive cohomology (for example P

is an homogeneous variety), then

$$H^n_{\mathrm{prim}}(X) \cong H^n_{\mathrm{van}}(X).$$

In this thesis we will therefore systematically use both terms in an interchangeable way, with distinction specified whenever relevant.

1.2 Griffiths residues for a projective hypersurface

As we said, Hodge theory provides a way to associate in a canonical way to a smooth projective variety X a bunch of complex vector spaces (together with an integral structure). The actual determination of these $H^{p,q}(X)$ turns out to be rather a difficult problem. The first and probably most important result in this context is Griffiths's description of the primitive cohomology ring of a smooth projective hypersurface in polynomial terms. Let us start by the motivating idea, that is the classical *adjunction formula.*

Theorem 1.2.1 (Adjunction Formula)**.** *Let P be a projective variety, X a smooth divisor in P. Then*

$$K_X = (K_P + X)|_X$$

where K is the canonical class (respectively of P and X). In a sheaf-theoretical language and taking global sections, $H^0(X, \omega_X) \cong H^0(X, \omega_P(X)|_X)$.

Now, let us consider the case of $P = \mathbb{P}^{n+1}$, with X a smooth hypersurface defined by the vanishing of a homogeneous polynomial of degree d, that is $X = V(f)$. Pick an element of $H^0(\mathbb{P}^{n+1}, \omega_{\mathbb{P}^{n+1}}(X))$, for instance

$$s = \frac{A\,\Omega_{\mathbb{P}^{n+1}}}{f}$$

where we can write locally $\Omega_{\mathbb{P}} = \mathrm{d}\,x_0 \wedge \ldots \wedge \mathrm{d}\,x_{n+1}$. Then up to a change of coordinates we can get an element of $H^0(X, \omega_X)$ by

$$\mathrm{Res}_X(s) = (-1)^{i-1}\,\frac{A\,\Omega^i}{f_i}$$

where $f_i := \frac{\partial f}{\partial x_i}$ and $\Omega^i := \mathrm{d}\,x_0 \wedge \ldots \wedge \widehat{\mathrm{d}\,x_i} \wedge \ldots \wedge \mathrm{d}\,x_{n+1},$. By smoothness we assume $f_i \neq 0$ for some i. This map is often called *Poincaré Residue.*

Using standard identifications the global sections of ω_X are determined uniquely by the elements of $H^0(\mathbb{P}^{n+1}, K_{\mathbb{P}^{n+1}}(X))$, and correspond to the possible choices for A, that has to be of degree $d-n-2$. So

$$H^0(X, \omega_X) \cong \mathbb{C}[x_0, \dots, x_n]_{d-n-2}.$$

We have seen that through adjunction formula we are able to detect $H^{n,0}(X)$. Is then natural to ask if even the other Hodge groups corresponds to some homogeneous component of a certain graded ring. The (positive) answer was first detected by Griffiths. From the analytic viewpoint, one raises the pole order and sends

$$s_p = \frac{A\,\omega_X}{f^p}$$

to

$$\operatorname{Res}_X(s_p) = \frac{A\,\omega^{i_1,\dots,i_p}}{f_{i_1}\dots f_{i_p}},$$

with notations generalising the previous one (e.g. at the denominator we consider the product of partial derivatives).
In the end we have (see for example [35] for an historical perspective on the subject)

Theorem 1.2.2 (Residues, Griffiths)**.** *The n-dimensional primitive Hodge structure of a smooth projective hypersurface $X \subset \mathbb{P}^{n+1}$ is given by the isomorphism*

$$H^{n-p+1,p-1}_{\text{prim}}(X) \cong (\mathbb{C}[x_0, \dots, x_{n+1}]/J_f)_{pd-n-2}$$

where J_f is the ideal (so-called Jacobian) spanned by all the partial derivatives $(f_0, \dots, f_n)$ of f.

The quotient $\frac{\mathbb{C}[x_0,\dots,x_{n+1}]}{J_f}$ is called the Jacobian ring (or algebra) R_f and it is quite relevant also in different areas of mathematics (for an example singularity theory). If there is no danger of confusion, we will often drop the pedix f and write simply R for R_f. Since the partial derivatives of a smooth hypersurfaces forms a regular sequence, the Hilbert-Poincaré series is particulary easy to describe, this being for $X \subset \mathbb{P}^n$ has degree d

$$\mathrm{HP}(R) = \frac{(1-t^{d-1})^{n+2}}{(1-t)^{n+2}}.$$

The above theorem generalise to the case of quasi smooth hypersurfaces in weighted projective spaces. The precise definition of quasi-smoothness is

Definition 1.2.3. X in $\mathbb{P}(a_0, \dots, a_n)$ is *quasi-smooth* of dimension m if its affine cone

C_X is smooth of dimension $m+1$ outside its vertex 0.

When $X \subset \mathbb{P}(a_0,\dots,a_n)$ is quasi-smooth, its singularities are induced by the group action, and hence are cyclic quotient singularities of type $\frac{1}{a_i}(a_0,\dots,\hat{a}_i,\dots,a_n)$. On a quasi-smooth variety X is it possible to define the notion of a pure Hodge structure. The construction is done by Dolgachev in all details in [54]. Consider in fact the smooth locus $j : X_0 \hookrightarrow X$ and $\widehat{\Omega}^p_X := j_* \Omega^p_{X_0}$. Then we can define $H^{p,q}(X)$ as in the smooth case and moreover $H^{p,q}(X) \cong H^q(X, \widehat{\Omega}^p_X)$. The Hodge decomposition takes then the form

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^q(X, \widehat{\Omega}^p_X).$$

Since there will be no danger of confusion, to avoide cumbersome notations when dealing with quasi-smooth varieties we will abuse the notation and write directly Ω^p_X instead of $\widehat{\Omega}^p_X$. In Dolgachev's paper is shown how to generalise the Residue theorem to this situation: more precisely we have

Proposition 1.2.4 (Residues for WPS). *Let X_d quasi-smooth in $\mathbb{P}(a_0,\dots,a_n)$. Then the Jacobian ring R is $\mathbb{C}[x_0,\dots,x_n]/J_f$, where x_i has degree a_i. In this case the quotient is finite-dimensional, and we have an isomorphism*

$$H^{n-p,p-1}_{\text{prim}}(X) = R_{pd-\sum a_i}$$

Moreover, the Hilbert Series of the Jacobian ring is given by the quotient

$$\text{HP}(R) = \frac{(1-t^{b_1})\dots(1-t^{b_n})}{(1-t^{a_1})\dots(1-t^{a_n})}$$

where $b_i := d - a_i$.

1.2.1 Generalisation to the complete intersection case

Griffiths calculus has been proved to be tremenduosly effective, as we are going to mention in the next section. Nevertheless, very few generalisations have been succesful so far. The most relevant one is probably the case of smooth complete intersection in $\mathbb{P}^n$. This is the work of several people in the beginning of 1990s, such as Dimca, Konno, Terasoma and many others. Key references are [53], [77]. The idea is to use a standard technique in projective geometry, namely the *Cayley trick*. Starting from Z a complete intersection in $\mathbb{P}^N$ one can construct an hypersurface $\widehat{Z}$ in a projective bundle $Y = \mathbb{P}(\bigoplus^c \mathcal{O}(d_i))$ over $\mathbb{P}^N$.

Denote by $\mathcal{L} = \mathcal{O}_Y(1)$ the (ample) dual of the tautological line bundle on the projective bundle Y. Call $\mathcal{E} = \bigoplus^c \mathcal{O}(d_i)$. One has $H^0(X, \mathcal{E}) \cong H^0(Y, \mathcal{L})$. Explicitely, take U an open subset of X over which $\mathcal{E}$ is trivial nd let $e_1, \dots, e_r$ a frame of $\mathcal{E}$ on U. If $\sigma \in H^0(X, \mathcal{E})$ is given locally by $\sigma = \sum \sigma_i e_i$, then the section $\hat{\sigma} = \sum_i \sigma_i e_i$, where we regard e's s homogeneous fiber coordinates on $\mathbb{P}(\mathcal{E}) \cong U \times \mathbb{P}^{c-1}$. Let Z and $\widehat{Z}$ the zero varieties of σ and $\hat{\sigma}$. Note that $\widehat{Z} \in H^0(Y, \mathcal{L})$. The Hodge theory of Z and $\widehat{Z}$ are strongly related: namely we have the following result

Proposition 1.2.5 (Proposition 4.3, [77])**.** *There is a canonical isomorphism of Hodge structures*

$$H^q_{\mathrm{van}}(Z, \mathbb{C})(1-c) \cong H^{q+2c-2}_{\mathrm{van}}(\widehat{Z}, \mathbb{C}).$$

From the isomorphism $H^0(X, \mathcal{E}) \cong H^0(Y, \mathcal{L})$ we can consider the total coordinate ring of $\widehat{Z}$ as

$$\mathcal{S} = \mathbb{C}[x_0, \dots, x_N, y_0, \dots, y_c].$$

The Picard group of $\widehat{Z}$ has rank two: therefore the ring above comes with a suitable bigrading. We have $\deg(x_i) = (0, 1)$ and $\deg(y_i) = (1, -d_i)$. The reason for this choice of bigrading is evident from the above isomorphism with the global section of the normal bundle $\mathcal{E}$. The result is then

Theorem 1.2.6 (Theorem 7 in [53])**.** *Let $Z = V(f_1, \dots, f_c)$ a smooth complete intersection of dimension n in $\mathbb{P}^N$ with normal bundle $\mathcal{E} = \bigoplus^c \mathcal{O}(d_i)$ and $\omega_Z \cong \mathcal{O}_Z(m)$. Let $F = \sum f_i y_i$. Denote by*

$$\mathcal{U}_{a,b} = (S/J)_{a,b}$$

where J is the ideal generated by $(\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial y_c})$. Then

$$\mathcal{U}_{p,m} \cong H^{n-p,p}_{\mathrm{prim}}(X).$$

We will return on Dimca's approach towards the end of this thesis, since it will be effective even in the context of homogeneous spaces.

1.3 Introduction to Torelli problem

In the previous section we described how to associate to a smooth projective variety X its Hodge structure. A very natural question to ask is how much informations on X are we losing by doing this. Therefore Torelli problem could be loosely stated as

Question 1.3.1 (Torelli)**.** *Is the Hodge structure associated to X enough to reconstruct X?*

Let us describe the problem in a more formal way. Let $X = X_0$ a smooth projective variety considered as the central fiber of a family $\pi : \mathcal{X} \to B$ over a smooth base. Up to restricting B, Ehresmann's theorem gives us canonical identifications for any k, for any $b \in B$

$$H^k(X_b, \mathbb{C}) \cong H^k(X_0, \mathbb{C}).$$

One can then make the following definition. Denote by $f^{p,k} := \dim F^p H^k(X, \mathbb{C})$.

Definition 1.3.2. The $p-$period map

$$\wp_{p,k} : B \to \mathrm{Gr}(f^{p,k}, H^k(X, \mathbb{C}))$$

is a map which to $b \in B$ associates the subspace

$$F^p(H^k(X_b), \mathbb{C}) \subset H^k(X_b, \mathbb{C}) \cong H^k(X, \mathbb{C}).$$

We define the *period map* $\wp^k$ as the k-tuple determined by $\wp^k := (\wp^{1,k}, \ldots, \wp^{k,k})$. If the index k is clear from the context, we will suppress it and denote the period map simply with $\wp$.

The global Torelli problem asks if the period map $\wp$ is an isomorphism. The question can be relaxed: for example the *generic* version of the Torelli theorem asks if the period map is of degree 1 on its image, or the *infinitesimal* Torelli problem, that asks if the differential of the period map, that can be computed as

$$\mathrm{d}\,\wp : H^1(T_X) \to \bigoplus_{p=0}^{n} \mathrm{Hom}(H^{p,q}(X), H^{p-1,q+1}(X))$$

is injective. The problem takes many avatars (mainly for historical reasons) according to which class of varieties are considered. For example let C, C' be two smooth projective curves, with associated Jacobians $J(C), J(C')$ and theta divisors Θ_C, $\Theta_{C'}$. Then

Theorem 1.3.3 (Torelli for curves)**.** *Let C, C' be two smooth genus g curves such that $(J(C), \Theta_C) \cong (J(C'), \Theta_{C'})$, then C and C' are isomorphic.*

One of the most famous Torelli-type theorem is the version for K3 surfaces. Recall that the groups $H^2(X, \mathbb{Z})$ forms a lattice, and it is therefore possible to speak about integral isometries. The theorem states

Theorem 1.3.4 (Torelli for K3 surfaces)**.** *Two K3 surfaces X and X' are isomorphic if and only if there exists a Hodge isometry $H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$.*

It is worth to mention that there exists both examples and counterexamples to all the versions of Torelli theorem. One of the most famous counterexamples to the Global Torelli problem is for Calabi-Yau threefolds (Balázs Szendröi's PhD thesis). In the case of hypersurfaces the explicit description of the Hodge structure given by residues has made possible a detailed analysis of the problem. We recap the key results in the following section.

1.3.1 Torelli problem for hypersurfaces

Griffiths description and Torelli for hypersurfaces in standard projective spaces Griffiths description of the primitive cohomology for a smooth hypersurface is a powerful tool to attack the Torelli problem. The problem was originally solved by Donagi, cf. [55]. A modern survey can be found, for example, in Claire Voisin's book [116], that here we recall briefly.

Assume that X_d is a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$. Assume that the dimension of X is at least 3 (in particular, all the deformations of X are projective). Thanks to the Lefschetz hyperplane the only interesting part of the Hodge structure of X is located in degree n. The starting point is realising how the differential of the period map for an Hodge structure of weight n restricted to its primitive subspaces

$$\mathrm{d}\,\wp : H^1(T_X) \to \bigoplus_{p=0}^{n} \mathrm{Hom}(H^{p,q}_{\mathrm{prim}}(X), H^{p-1,q+1}_{\mathrm{prim}}(X)),$$

can be rewritten thanks to the Griffiths description as

$$R_d \longrightarrow \bigoplus_{p=0}^{n} \mathrm{Hom}(R_{(p+1)d-n-2}, R_{(p+2)d-n-2}),$$

where R denote as usual the Jacobian ring and the subscript refers to its homogeneous components.
We can rephrase this using local duality theorem (Theorem 2.2 of [115]). First recall the concept of *socle*. For a graded k-Algebra A, the socle is defined as

$$\mathrm{Soc}(A) = \{h \in A \mid hg = 0 \text{ for all } g \in \bigoplus A_i\}.$$

In general the socle could be either empty or in different degrees, but in case of the

Jacobian ring it coincides with a specific degree component R_σ, cf. Corollary A3, [115]. One has in particular $R_{\sigma+j} = 0$ for every $j > 0$. The local duality theorem is then

Theorem 1.3.5 (Theorem 2.2 in [115])**.** *Let $f_0, \ldots, f_{n+1}$ a regular sequence of weighted homogeneous polyonomial in $\mathbb{C}[x_0, \ldots, x_{n+1}]$ and let*

$$A = \mathbb{C}[x_0, \ldots, x_{n+1}]/(f_0, \ldots, f_{n+1}).$$

Suppose a_i is the weight of x_i and d_i = deg f_i. Then for any $0 \leq a \leq \sigma$ the pairing given by multiplication

$$A_a \times A_{\sigma-a} \longrightarrow A_\sigma$$

is non-degenerate, where $\sigma = \sum d_i - a_i$ is the top degree.

If we pick $A = R$, the Jacobian ring, the theorem applies since the partial derivatives forms a regular sequence.
The above theorem allows us to check, instead of the injectivity of

$$R_d \longrightarrow \operatorname{Hom}(R_a, R_b)$$

the surjectivity of

$$R_b \times R_{\sigma-(a+b)} \to R_{\sigma-a}.$$

However, the above theorem assures the non degeneracy of the multiplication map only when the socle is involved.
The non degeneracy of the general multiplication map

$$R_a \times R_b \longrightarrow R_{a+b}$$

is indeed tackled by *Macaulay's theorem.*

Theorem 1.3.6 (Theorem 1 in [115])**.** *Let $f_0, \ldots, f_{n+1}$ be a regular sequence of homogeneous polynomials of degree $d_0, \ldots, d_{n+1}$ in $\mathbb{C}[x_0, \ldots, x_{n+1}]$ and let*

$$R = \mathbb{C}[x_0, \ldots, x_{n+1}]/(f_0, \ldots, f_{n+1})$$

. Then R is a finite dimensional graded $\mathbb{C}$-algebra with top degree $\sigma = \sum(d_i - 1)$ and the multiplication map

$$\mu : R_a \times R_b \longrightarrow R_{a+b}$$

is nondegenerate for $a + b \leq \sigma$.

In turn this is enough to guarantee the infinitesimal Torelli theorem for smooth projective hypersurfaces in $\mathbb{P}^{n+1}$.

The situation becomes more interesting when we consider the *generic* Torelli problem. This is the question of knowing whether the period map $\wp$ is degree 1 over its image. In other words, given two hypersurfaces Y and Y' of degree d in $\mathbb{P}^{n+1}$ with Y generic such that there is an isomorphism of polarised Hodge structures

$$i : H^n_{\mathrm{prim}}(Y, \mathbb{Z}) \cong H^n_{\mathrm{prim}}(Y', \mathbb{Z})$$

one asks wheter Y and Y' are isomorphic. Using Griffiths technique, Donagi first and Cox and Green subsequently [1] proved

Theorem 1.3.7 (Theorem 6.4 in [55], Theorem 1 in [44])**.** *The generic Torelli theorem holds for hypersurfaces of degree d in $\mathbb{P}^{n+1}$, up to the possible exceptions of the following cases*

1. *d divides $n + 2$;*
2. *$d = 3,\ n + 1 = 3$;*
3. *$d = 4,\ n + 1 \equiv 1$ mod 4;*

Case (ii) represents a true exception. In case (i) the answer is expected to be positive up to a finite number of counterexamples. Cases known are cubic curves in $\mathbb{P}^2$, quartic surfaces in $\mathbb{P}^3$ and quintics in $\mathbb{P}^5$.

Infinitesimal Torelli for weighted hypersurfaces A natural extension of the Donagi work is consider the case of quasi-smooth hypersurfaces in weighted projective spaces. Let alone the generic Torelli, one fails to get a general answer even for the infinitesimal Torelli.

The main obstacle is that the weighted version of Macaulay's theorem does not hold for any weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ (more in general, for any weighted graded rings). Counterexamples can be easily given. Consider for example the case of $R = \mathbb{C}[x, y]/(x^2, y^3)$. The multiplication map $\mu : R_1 \times R_3 \to R_4$ is degenerate, since $x_0 \cdot R_3 = 0$ with $x_0 \neq 0$.

Nevertheless on some weighted graded rings some generalisations of Macaulay's theorem hold. Based on a detailed analysis, Tu was able to identify in [115] several class

[1] Cox and Green refined Donagi's argument, solving the case $d = 6,\ n \equiv 2$ mod 6 left open by Donagi.

of quasi-smooth hypersurfaces for which the infinitesimal Torelli theorem actually hold. We recap briefly here his results.
For $w\mathbb{P} = \mathbb{P}(a_0, \ldots, a_{n+1})$ set

$$s = \sum a_i, \quad m = lcm(a_0, \ldots, a_{n+1})$$

and for any subset $J = (j_1, \ldots, j_n)$ of $\{0, \ldots, n+1\}$ we define

$$m(a|J) := lcm(a_{j_1}, \ldots, a_{j_n}).$$

We define

$$G = -s + \frac{1}{n+1} \sum_{2 \leq k \leq n+2} \binom{n}{k-2}^{-1} \sum_{|J|=k} m(a|J).$$

An estimate for G is

$$G \leq -s + m(n+1);$$

in particular we notice that for the standard projective space $\mathbb{P}^{n+1}$ we have $s = n+2, m = 1, G = -1$. What we have is

Theorem 1.3.8 (Theorem 2.8 in [115])**.** *Let $R = \mathbb{C}[x_0, \ldots, x_{n+1}]/J$ be the weighted ring defined by the ideal J of a regular sequence $f_0, \ldots, f_{n+1}$. Set $d_i = \deg f_i$, $a_i =$ weight x_i and $\sigma = \sum(d_i - a_i)$. The natural map*

$$R_a \to \mathrm{Hom}(R_b, R_{a+b})$$

is injective

1. *if b is a multiple of m and $\sigma - (a+b) \geq max(G+1, 0)$, or*
2. *if $\sigma - (a+b)$ is a multiple of m and $b \geq G+1$.*

In the next two results let m and s be as above. The following holds:

Theorem 1.3.9 (Theorem 2.10 in [115])**.** *Let p an integer between 1 and n for which gcd(m,p) divides s. Then there are infinitely many nonnegative integers $k \geq ((n+1)p/(n+1-p)) - (s/m)$ for which $d = (s+km)/p$ is a positive integer. The infinitesimal Torelli theorem holds for quasi-smooth hypersurfaces of degree d in $\mathbb{P}(a_0, \ldots, a_{n+1})$.*

For other specific choices of the weights it is known:

Theorem 1.3.10 (Theorem 4.1 in [57]). *Let $\mathbb{P}$ a weighted projective space $\mathbb{P}(a_0, \ldots, a_{n+1})$ for which $a_0 = a_1 = 1$ and m divides s, and let $\mathcal{M}$ the space of quasi-smooth weighted hypersurfaces of degree d in $\mathbb{P}$ modulo automorphisms of $\mathbb{P}$. Assume d is a multiple of m and $d \geq max(3s, s + m(n+1))$. Then there is an open dense subset of $\mathcal{M}$ on which the period map is defined and injective.*

Many interesting cases are still left open. In particular the answer was not known for $\mathbb{Q}$-Fano threefolds hypersurfaces, despite their importance in terms of birational geometry. In the first chapter of this thesis we give an answer to this problem, following a careful analysis of the Jacobian rings involved.

Chapter 2

Infinitesimal Torelli problem for $\mathbb{Q}$-Fano threefold hypersurfaces

In this chapter we give a full answer to the infinitesimal Torelli problem in the case of quasi-smooth $\mathbb{Q}$-Fano hypersurfaces of dimension 3 with terminal singularities and with Picard number 1. We introduce then an example of a rather straightforward geometrical procedure, that produces a curious periodicity behaviour in Hodge theory.

2.1 Fano threefold hypersurfaces and Torelli problem

As a first application of Griffiths theory we solve the infinitesimal Torelli problem in the case of $\mathbb{Q}$-Fano Threefolds.
Recall that in general a $\mathbb{Q}$-Fano variety is a projective variety X such that X has at worst $\mathbb{Q}$-factorial terminal singularities, ω_X^{-1} is ample and $\mathrm{Pic}(X)$ has rank 1; cf. [40]. We focus here on the threefolds case, probably the most relevant from an historical perspective. The index 1 list was first discovered by Iano-Fletcher and Reid. In the higher index case, we mention important contributions by Brown, Prokhorov, Reid, Suzuki, Takagi, et.al; cf. [94], [28], [113]. A convenient way to visualise the list of all this Fano with their birational invariants is provided by the graded ring database, [24].
Denote with ι_X the *index* of X, that is $\omega_X \cong \mathcal{O}_X(-\iota_X)$. If we restrict to the codimension 1 case, we have 95 families [1] with $\iota_X = 1$ and 35 families with $\iota_X > 1$. By an abuse of notation, we denote by X both the family and a generic quasi-smooth member. We follow the common approach and divide our analysis in this two cases, starting from the index 1 one. We do not list here all the 95 families of index 1. The interested reader can

[1]These families are often quoted in literature as "the famous 95".

indeed look at the original paper of Iano-Fletcher ([68]) or consult the online database.

2.1.1 Infinitesimal Torelli for Fano varieties of index 1

Recall from the introduction the Griffiths description of the differential of the period map for an Hodge structure of weight k

$$\mathrm{d}\,\wp : H^1(T_X) \to \bigoplus_{p=0}^{k} \mathrm{Hom}(H^{p,q}(X), H^{p-1,q+1}(X)).$$

In particular, its injectivity is enough for the infinitesimal Torelli theorem to hold. Assume that X_d is a smooth hypersurface of degree d in $w\mathbb{P}^n(a_0, \ldots, a_n)$ of dimension at least 3, and that $H^2(\mathcal{O}_X) = 0$ (equivalently, all the deformations of X are projective). Denote by $s = \sum a_i$. The Griffiths-Steenbrink description of the primitive cohomology of a quasi-smooth hypersurface reduces the problem to the injectivity of the polynomial map

$$R_d \longrightarrow \bigoplus_{p=0}^{n} \mathrm{Hom}(R_{(p+1)d-s}, R_{(p+2)d-s}),$$

where R denote as usual the Jacobian ring and the subscript refers to its homogeneous components.

For a quasi-smooth Fano threefolds we have $H^{3,0}(X) = H^{0,3}(X) = 0$: moreover if we focus on the index 1 condition what we have to verify in order to have the infinitesimal Torelli for any of the previous 95 families is to check the injectivity of the natural map

$$R_d \to \mathrm{Hom}(R_{d-1}, R_{2d-1}).$$

If we use local duality theorem 1.3.5, since the socle coincides with the homogeneous component in degree $\sigma = 3d - 2\iota_X$ we can rephrase the problem as following

Remark 2.1.1. Let X as above. The infinitesimal Torelli theorem holds for X if the natural map

$$\mathrm{Sym}^2(R_{d-1}) \to R_{2d-2}$$

is surjective.

We first try to check the condition of weighted Macaulay's theorem recalled in the introduction. Consider for example the family no.5, that is $X_7 \subset \mathbb{P}(1,1,1,2,3)$. Using the notations of Theorem 1.3.9, the equation that need to be satisfied is $d = 7 = (8+6k)/p$, together with the condition $k \geq (4p/4 - p) - 4/3$, but this is clearly not possible. By similar standard computations we get

Lemma 2.1.2. *Only the families no.1 and no.2 of the Fletcher-Reid list satisfies the numerical conditions of the weighted Macaulay's theorem. These are* $X_4 \subset \mathbb{P}^4$ *and* $X_5 \subset \mathbb{P}(1^4, 2)$*.*

In the other cases, a partial answer can be obtained by looking at the surjectivity of the multiplication map in the ambient ring. More precisely

Proposition 2.1.3 (Proposition 2.3 in [115])**.** *Denote by* $S = \mathbb{C}[x_0, \dots, x_n]$*. Let* $R = \mathbb{C}[x_0, \dots, x_n]/J$ *be a weighted ring for which local duality holds, and let* σ *be the top degree of* R*. Given non-negative integers* a, b *satisfying* $a + b \leq \sigma$*, if*

$$S_b \times S_{\sigma-(a+b)} \to S_{\sigma-a}$$

is surjective, then $R_a \to \mathrm{Hom}(R_b, R_{a+b})$ *is injective.*

In our case, we have $a = d, b = d-1$ and $\sigma = \sum d - 2a_i = 5d - 2s$, with $s = d+1$. Therefore we have to check the result for $S_{d-1} \times S_{d-1} \to S_{2d-2}$, or equivalently the surjectivity of the natural map

$$\mathrm{Sym}^2(S_{d-1}) \to S_{2d-2}.$$

Once we checked the result, we have the following

Lemma 2.1.4. *Of the* 93 *remaining families, only the no.* 5*, that is* $X_7 \subset \mathbb{P}(1, 1, 1, 2, 3)$ *satisfies the surjectivity already at the level of polynomial ring.*

Proof. On X_7 the multiplication map $\mathrm{Sym}^2(S_6) \to S_{12}$ can be verified to be surjective. All the other cases yields counterexamples. For example consider $X_6 \subset \mathbb{P}(1^4, 3)$. $\mathrm{Sym}^2(S_5) \to S_{10}$ is not surjective as we can see by the element $x_4^3 x_0$, where x_4 is the variable of weights 3. Indeed is clear that $S_5 = \langle \mathrm{Sym}^5(x_0, \dots, x_3), \mathrm{Sym}^2(x_0, \dots, x_3) \cdot x_4 \rangle$. An extensive computer search using Macaulay2 confirms this statement. $\square$

Corollary 2.1.5. *Let* X *any quasi smooth member of the family* $X_7 \subset \mathbb{P}(1, 1, 1, 2, 3)$*. Then the differential of the Period map*

$$\mathrm{d}\,\wp : H^1(X, T_X) \to \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X))$$

is injective, and therefore the local Torelli theorem holds.

To prove the theorem for the other cases, we have to use the following lemma

Lemma 2.1.6. *Let $\pi : \mathcal{X} \to U$ be a flat family of quasi-smooth hypersurfaces such that for the central fiber $X_0 = \pi^{-1}(0)$ the infinitesimal Torelli property holds. Then the same holds for the fibers $\pi^{-1}(U)$ over an open $0 \in U$.*

Proof. We recall that $\mathcal{M}_0$ is a complex manifold and the period domain is a variety. Hence the condition of the differential of the period map being of maximal rank is an open condition. $\square$

Theorem 2.1.7. *Let $X_d \subset \mathbb{P}(a_0, \dots, a_4)$ one of the 92 families left (not 1,2,5). Then the infinitesimal Torelli theorem holds generically for X_d.*

Proof. The idea of the proof is simple. We check the surjectivity of the dual of the infinitesimal period map on a quasi-smooth element X_0. A genericity condition can be found for example in [68]. Once this is confirmed, by the lemma 2.1.6 the result will hold in a Zariski open set $0 \in U$, where we can think of X_0 as central fiber of a flat family $\pi : \mathcal{X} \to U$.
The check on the central fiber can be done with the help of the following Macaulay2 code

```
def=(d, weights)->(
R=QQ[x,y,z,t,w, Degrees=>weights];
 f=random(d,R); J=Jacobian matrix f;
B=(R/ideal J);L1=flatten entries (symmetricPower(2, basis(d-1,B)));
L2=flatten entries basis(2*d-2,B);
APP=select(L2, a-> not member(a, L1));
IL=ideal(L1);  print APP, print LINCOMB,
for i in APP do if ((i % IL)!=0) then print i else print 0
)
```

Actually all the computations could be solved by hand in principle. To have a concrete grasp of how this work we refer to the example 2.1.1, where we deal with the (most) interesting case of index >1. An extensive computer search with Macaulay2 confirms that this holds for any of the 92 remaining families. $\square$

2.1.2 Infinitesimal Torelli for Fano varieties of index >1

We now investigate the case of higher index Fano threefolds. Unlike the index 1 case, here the situation is much more various and complicated: in particular we will have (other than the Hodge-trivial examples) some families for which the infinitesimal Torelli fails, and some families for which it holds. We recall in 2.1 below the list of Fano threefold of higher index, from Okada (see [89]), ordered according to the index.

No.	$X_d \subset \mathbb{P}(a_0, \ldots, a_4)$	Ind	Torelli	No.	$X_d \subset \mathbb{P}(a_0, \ldots, a_4)$	Ind	Torelli
96	$X_3 \subset \mathbb{P}(1,1,1,1,1)$	2	T	113	$X_4 \subset \mathbb{P}(1,1,2,2,3)$	5	R
97	$X_4 \subset \mathbb{P}(1,1,1,1,2)$	2	T	114	$X_6 \subset \mathbb{P}(1,1,2,3,4)$	5	T
98	$X_6 \subset \mathbb{P}(1,1,1,2,3)$	2	T	115	$X_6 \subset \mathbb{P}(1,2,2,3,3)$	5	AT
99	$X_{10} \subset \mathbb{P}(1,1,2,3,5)$	2	T	116	$X_{10} \subset \mathbb{P}(1,2,3,4,5)$	5	T
100	$X_{18} \subset \mathbb{P}(1,2,3,5,9)$	2	T	117	$X_{15} \subset \mathbb{P}(1,3,4,5,7)$	5	T
101	$X_{22} \subset \mathbb{P}(1,2,3,7,11)$	2	T	118	$X_6 \subset \mathbb{P}(1,1,2,3,5)$	6	T
102	$X_{26} \subset \mathbb{P}(1,2,5,7,13)$	2	T	119	$X_6 \subset \mathbb{P}(1,2,2,3,5)$	7	R
103	$X_{38} \subset \mathbb{P}(2,3,5,11,19)$	2	T	120	$X_6 \subset \mathbb{P}(1,2,3,3,4)$	7	R
104	$X_2 \subset \mathbb{P}(1,1,1,1,1)$	3	R	121	$X_8 \subset \mathbb{P}(1,2,3,4,5)$	7	AT
105	$X_3 \subset \mathbb{P}(1,1,1,1,2)$	3	T	122	$X_{14} \subset \mathbb{P}(2,3,4,5,7)$	7	AT
106	$X_4 \subset \mathbb{P}(1,1,1,2,2)$	3	T	123	$X_6 \subset \mathbb{P}(1,2,3,3,5)$	8	R
107	$X_6 \subset \mathbb{P}(1,1,2,2,3)$	3	T	124	$X_{10} \subset \mathbb{P}(1,2,3,5,7)$	8	T
108	$X_{12} \subset \mathbb{P}(1,2,3,4,5)$	3	T	125	$X_{12} \subset \mathbb{P}(1,3,4,5,7)$	8	T
109	$X_{15} \subset \mathbb{P}(1,2,3,5,7)$	3	T	126	$X_6 \subset \mathbb{P}(1,2,3,4,5)$	9	R
110	$X_{21} \subset \mathbb{P}(1,3,5,7,8)$	3	T	127	$X_{12} \subset \mathbb{P}(2,3,4,5,7)$	9	AT
111	$X_4 \subset \mathbb{P}(1,1,1,2,3)$	4	T	128	$X_{12} \subset \mathbb{P}(1,4,5,6,7)$	11	T
112	$X_6 \subset \mathbb{P}(1,1,2,3,3)$	4	T	129	$X_{10} \subset \mathbb{P}(2,3,4,5,7)$	11	R
				130	$X_{12} \subset \mathbb{P}(3,4,5,6,7)$	13	R

Table 2.1: List of Fano threefolds of index 1 and their behaviour with respect to Torelli problem. The notation AT/T/R stands for (respectively) anti-Torelli, Torelli, rigid, as explained in the following pages.

What do we have to check? Keeping notations as in the introduction we have

Lemma 2.1.8. *Let $X_d \subset \mathbb{P}(a_0, \ldots, a_4)$ a quasi-smooth Fano threefold hypersurface of index ι_X. Then the infinitesimal Torelli theorem holds if the natural map*

$$R_{d-\iota_X} \times R_{d-\iota_X} \to R_{2d-2\iota_X}$$

is surjective.

Proof. Recall that we have to check to the injectivity of the map

$$R_d \to \mathrm{Hom}(R_{d-\iota_X}, R_{2d-\iota_X}),$$

where $R_{d-\iota_X} \cong H^{2,1}(X)$ and $R_{2d-\iota_X} \cong H^{1,2}(X) \cong (H^{2,1}(X))^\vee$ by Serre duality. Equivalently, using local duality theorem (Theorem 2.2 of [115]) what we need to verify is the surjectivity of the natural multiplication map

$$R_{d-\iota_X} \times R_{\sigma-(2d-\iota_X)} \to R_{\sigma-d}.$$

On the other hand, since $s = \sum a_i = d + \iota_X$, we have $\sigma = 5d - 2s = 3d - 2\iota_X$. □

We will now analyse separately the three interesting cases, that is Hodge-rigid families, the anti-Torelli and the Torelli.

Hodge-rigid families We call an $X_d \subset w\mathbb{P}$ *Hodge-rigid* if both $H^1(X, T_X) = 0$ and $H^{2,1}(X) = 0$. Now, for these threefolds we do not have any Torelli-type question to ask: therefore we want to classify and remove these cases from our list. Now, we recall that from Griffiths-Steenbrink theory we have $R_d \cong H^1(X, T_X)$, and by Serre duality

$$H^1(X, \Omega_X^2) \cong H^1(T_X \otimes \omega_X) = H^1(X, T_X(-\iota_X)) = R_{d-\iota_X}.$$

From this it clearly follows that if $d < \iota_X$, then $H^{2,1}(X) = 0$.

Lemma 2.1.9. *The following families satisfies $H^{2,1}(X) = H^1(X, T_X) = 0$. Therefore they are Hodge-rigid.*

- no. 104 $X_2 \subset \mathbb{P}^4$;
- no. 113 $X_4 \subset \mathbb{P}(1,1,2,2,3)$;
- no. 119 $X_6 \subset \mathbb{P}(1,2,2,3,5)$;
- no. 120 $X_6 \subset \mathbb{P}(1,2,3,3,4)$;
- no. 123 $X_6 \subset \mathbb{P}(1,2,3,3,5)$;
- no. 126 $X_6 \subset \mathbb{P}(1,2,3,4,5)$;
- no. 129 $X_{10} \subset \mathbb{P}(2,3,4,5,7)$;
- no. 130 $X_{12} \subset \mathbb{P}(3,4,5,6,7)$.

Proof. The vanishing of $H^{2,1}(X)$ is assured by the condition $d < \iota_X$ above. To verify the vanishing of $H^1(X, T_X)$, simply notice that for every member of the list above one has $d > \sigma$. Therefore, by definition of socle, $R_d = 0$. □

2.1.3 Torelli and anti-Torelli families

We investigate first the families that do not satisfy the infinitesimal Torelli property.

Theorem 2.1.10. *Let X_d any of the four families* no. 115, 121, 122, 127. Then the infinitesimal Torelli does not hold for X_d.

Proof. We will analyse separately the four different cases.

no. 115. Pick any general quasi smooth member of the family of index $\iota_X = 5$, $X_6 \subset \mathbb{P}(1,2,2,3,3)$, where we name coordinates x, y_0, y_1, z_0, z_1. Since $d - \iota_X = 1$ we have to check the surjectivity of the map: $R_1 \times R_1 \to R_2$. Now, since the partial derivatives form a regular sequence, if we compute the Hilbert-Poincaré series of the Jacobian ring we have

$$\mathrm{HP}(R) = \prod \frac{(1 - t^{d-a_i})}{(1 - t^{a_i})} = 1 + t + 3t^2 + 3t^3 + 4t^4 + 3t^5 + 3t^6 + t^7 + t^8.$$

Therefore, since there is only one generator of degree one, we have $\mathrm{Sym}^2(R_1) \cong < x^2 >$, whereas R_2 contains, for example, y_0, y_1, and the natural map cannot be surjective. Dualising we have the standard map from R_d given by

$$\mathbb{C}^3 \cong R_6 \to \mathrm{Hom}(R_1, R_7) \cong \mathbb{C},$$

and this cannot be injective.

no. 121. The same phenomenon occurs for $X_8 \subset \mathbb{P}(1,2,3,4,5)$ of index 7, with coordinates x, y, z, v, w. The Hilbert-Poincaré series is

$$\mathrm{HP}(R) = 1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + 3t^6 + 2t^7 + 2t^8 + t^9 + t^{10}$$

and for $\mathbb{C}^2 \cong R_8 \to \mathrm{Hom}(R_1, R_9) \cong \mathbb{C}$ injectivity clearly fails.

no. 122. This is $X_{14} \subset \mathbb{P}(2,3,4,5,7)$, of index $\iota_X = 7$. Here we have as Hilbert-Poincaré series

$$\mathrm{HP}(R) = 1 + t^2 + t^3 + 2t^4 + 2t^5 + 3t^6 + 3t^7 + 5t^8 + 4t^9 + 6t^{10} +$$

$$+5t^{11} + 7t^{12} + 6t^{13} + 7t^{14} + 6t^{15} + 7t^{16} + 5t^{17} +$$

$$+6t^{18} + 4t^{19} + 5t^{20} + 3t^{21} + 3t^{22} + 2t^{23} + 2t^{24} + t^{25} + t^{26} + t^{28}.$$

Here we have $R_{14} \cong \mathbb{C}^7$ and $R_7 \cong \mathbb{C}^3$, so we cannot conclude immediately injectivity. Nevertheless, $\mathrm{Sym}^2(R_7)$ has dimension 6, so surjectivity is excluded.

no. 127. This is $X_{12} \subset \mathbb{P}(2,3,4,5,7)$. The Hilbert-Poincaré series is

$$\mathrm{HP}(R) = 1 + t^2 + t^3 + 2t^4 + t^5 + 3t^6 + 2t^7 + 3t^8 + 2t^9 + 3t^{10} + 2t^{11} + 3t^{12} + t^{13} +$$

$$+2t^{14} + t^{15} + t^{16} + t^{18}.$$

Thus $R_{12} \cong \mathbb{C}^3$, while $\mathrm{Hom}(R_3, R_{15}) \cong \mathbb{C}$.

$\square$

We analyse in a detailed example the failure of the Torelli problem in one of the above examples.

Failure for no.122. Here we use the following member of the family defined by the polynomial

$$f = x_0^7 + x_0 x_2^3 + x_1^3 x_3 + x_2 x_3^2 + x_4^2$$

where $V(f) \subset \mathbb{P}(2,3,4,5,7)$. Looking at the equation the Jacobian ideal has no zeroes.

Unlike the previous examples, here the non-injectivity cannot be deduced a priori, but follows from a careful examination of the multiplication map. In the following table we list the generators of the interesting graded component of the Jacobian ring R_f and the anti-Torelli deformations associated with f.

R_7	$\langle x_0^2 x_1, x_0 x_3, x_1 x_2 \rangle$
R_{21}	$\langle x_0^4 x_1 x_3^2, x_0^3 x_3^3, x_2^4 x_3 \rangle$
R_{14}	$\langle x_0 x_1 x_2 x_3, x_0^4 x_1^2, x_0^5 x_2, x_0^3 x_1 x_3, x_0^2 x_1 x_2, x_0^2 x_3^2, x_1^2 x_2^2 \rangle$
K_f	$\langle x_0 x_1 x_2 x_3 \rangle$

Table 2.2: Torelli and Anti-Torelli for family no.122

To check the kernel K_f we proceed as follows. First we check using the previous tools that the element of $R_{2d-2\iota}$ in the cokernel of multiplication map is $x_0^5 x_2$. Then we dualize via $(R_k)^\vee \cong R_{\sigma-k}$. Here $R_d = R_{14} = R_{\sigma-d}$. The (non-canonical) isomorphism between R_{14} and itself yielding duality pairs up $x_0^5 x_2$ with $\frac{1}{32} x_0 x_1 x_2 x_3$ with respect to the socle basis $x_0^5 x_1 x_3^3$. One can in fact verifies that the above multiplication gives exactly the socle generator, whereas any other multiplication of $x_0^5 x_2$ with any other basis element of R_{14} lies in J_f. Indeed one can identify $\frac{1}{32} x_0 x_1 x_2 x_3$ as dual element of $x_0^5 x_2$. Therefore a family of Anti-Torelli deformation with central fiber the fixed X_f will be given by $V(f + \lambda x_0 x_1 x_2 x_3)$

The Torelli families of index > 1

The other 24 families behave in the opposite way. We were able to check the surjectivity statement already at the level of the ring S, that is $\mathrm{Sym}^2 S_{d-\iota_X} \to S_{2d-2\iota_X}$, for the families no. 98, 111, 118, 128. This is enough to guarantee the surjectivity at the level of the Jacobian ring since $S_{2d-2\iota_X} \to R_{2d-2\iota_X}$ is the quotient map. No. 111 and No. 118 exhibits a curious behaviour. They are respectively $X_4 \subset \mathbb{P}(1,1,1,2,3)$ and $X_6 \subset \mathbb{P}(1,1,2,3,5)$. They both verifies $d = \iota_X$ and therefore $d = \sigma$. In particular, one has to check either the surjectivity of the natural map $R_0 \times R_0 \to R_0$ or the injectivity of $R_\sigma \to \mathrm{Hom}(R_0, R_\sigma)$, and this trivially holds. For all the other families we proceed by the means of an extensive computer search, as in the index 1 case. We give here an example.

Example 2.1.1. Consider $X_6 \subset \mathbb{P}(1,1,2,3,4)$ in the class no.114 given by the equation $x_0^6 + x_1^6 + x_2^3 + x_3^2 + x_2 x_4$. Note that X_6 is quasi-smooth. The Hilbert Series of its Jacobian ring is

$$\mathrm{HP}(R) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 4t^5 + 3t^6 + 2t^7 + t^8$$

We have to check the surjectivity $\mathrm{Sym}^2 R_1 \to R_2$. The space R_1 is generated by x_0, x_1, while R_2 is generated by $x_0^2, x_0 x_1, x_1^2$. The variable x_2 of weight 2 is in the Jacobian ideal. $\mathrm{Sym}^2 R_1$ is 3-dimensional, and equal to R_2. In particular the multiplication map is surjective. Therefore infinitesimal Torelli holds.

The same genericity argument of Lemma 2.1.6 yields

Theorem 2.1.11. *Let $\mathcal{M}$ the space of quasi-smooth weighted hypersurfaces of degree d in $\mathbb{P}$ modulo automorphisms of $\mathbb{P}$, for any of the non Hodge-trivial families of quasi-smooth Fano Threefolds of index $i_X > 1$. Then*

- *for the families no. 115, 121, 122 and 127 the infinitesimal Torelli theorem does not hold;*
- *for the remaining 24 families, there is an open dense subset of $\mathcal{M}$ on which the infinitesimal Torelli theorem holds.*

2.2 Periodic patterns in Hodge theory

Let us consider a weighted hypersurface $X_d^0 = V(f_0) \subset \mathbb{P}(a_0, \dots, a_n)$. Suppose that $d \equiv 0$ $(mod\ 2)$ and call $d = 2t$. Consider now

$$X_d^1 \xrightarrow{2:1} \mathbb{P}(a_0, \dots, a_n)$$

a double cover of the ambient weighted projective space, branched over X_d^0. Since d is even, X_d^1 has a model as

$$X_d^1 = V(f_0 + y_1^2) \subset \mathbb{P}(a_0, \dots, a_n, t).$$

Let us call $f_1 := f_0 + y_1^2$. Suppose now that $\omega_{X_d^0} \cong \mathcal{O}_{X_d^0}(m_0)$: by adjunction one has

$$\omega_{X_d^1} \cong \mathcal{O}_{X_d^1}(-\sum a_i - t + d) \cong \mathcal{O}_{X_d^1}(m_0 - t) =: \mathcal{O}_{X_d^1}(m_1).$$

We have the following immediate result

Lemma 2.2.1. *X_d^0 is quasi-smooth if and only if X_d^1 is.*

Proof. Simply notice that $J_{f_1} = (J_{f_0}, y_1)$. Therefore the only possible new singularities of X_d^1 comes just from the new ambient space. □

Set $R_{f_0} = S/(J_{f_0})$ and similarly R_{f_1} for X_d^1. It is clear that $R_{f_1} \cong S[y_1]/(J_{f_0}, y_1)$, therefore by the hyperplane section principle one has $R_{f_0} \cong R_{f_1}$. We pick any general member $X_d^1 = V(g) \subset \mathbb{P}(a_0, \dots, a_n, t)$. Since the partial derivatives form a regular sequence, one clearly has $\dim(R_{f_1})_k = \dim(R_g)_k$, for all k and we have an isomorphism of $\mathbb{C}$-vector spaces between the two Jacobian rings. Now we see that by completing the square the "double cover type" does not form a proper subfamily inside the space of all $X_d^1 \subset \mathbb{P}(a_0, \dots, a_n, t)$. Clearly the process can go on to give an infinite chain of double covers

$$\begin{tikzcd}
 & & & \cdots \arrow[d, "\varphi_3"] \\
 & & X_d^2 \arrow[r, hook] \arrow[d, "\varphi_2"] & w\mathbb{P}(\underline{a}, t, t) \\
 & X_d^1 \arrow[r, hook] \arrow[d, "\varphi_1"] & w\mathbb{P}(\underline{a}, t) & \\
X_d^0 \arrow[r, hook] & w\mathbb{P}(\underline{a}) & &
\end{tikzcd} \tag{2.1}$$

where any X_d^j is a double cover of the projective space $\mathbb{P}(a_0, \dots, a_n, t^{j-1})$ branched on X_d^{j-1}. By hyperplane section principle (see [101])

Proposition 2.2.2. *For any X_d^j obtained with tower construction one has $R_{f_j} \cong R_{f_0}$.*

Let us call an *even member* of the tower an X_d^{2k} obtained by doing an even number of step in the construction above. Similarly we will define the *odd members*.

Proposition 2.2.3.

- *Let X_d^{2k} any even member of the tower, of dimension $n+2k$. We have that*

$$H^{n+2k}(X_d^{2k}, \mathbb{C}) \cong H^n(X_d^0, \mathbb{C}).$$

The isomorphism is compatible with the Hodge decomposition: in particular the central Hodge numbers of X_d^0 are the same of the Hodge numbers of X_d^{2k} up to a degree k shift, that is

$$\left(h^{n+2k,0}_{X_d^{2k}}, h^{n+2k-1,1}_{X_d^{2k}}, \dots, h^{1,n+2k-1}_{X_d^{2k}}, h^{0,n+2k}_{X_d^{2k}}\right) = \left(0, \dots, 0, h^{n,0}_{X_d^0}, \dots, h^{0,n}_{X_d^0}, 0, \dots, 0\right),$$

with $2k$ zeroes on the last vector;

- *the same holds for odd members, with an equality between the Hodge numbers of X_d^1 and X_d^{2k+1}, for any k.*

Proof. This is just a careful analysis of the involved components of the Jacobian ring. Let us start from the base of our chain, $X_d^0 \subset \mathbb{P}(a_0, \dots, a_n)$. We will denote as before

$d = 2t$ and $s = \sum a_i$ Since by Proposition 7.1.2 the Jacobian ring is the same in any step of our construction, we drop the subscript referring to the equation and denote it simply with R. By Griffiths-Steenbrink one has

$$H^{n-p,p}_{\text{prim}}(X^0_d) \cong R_{(p+1)d-s}$$

Now extend to $X^d_1 \subset \mathbb{P}(a_0, \dots, a_n, t)$. By the same argument we have

$$H^{(n+1)-p,p}_{\text{prim}}(X^1_d) \cong R_{(p+1)d-s-t}.$$

If we extend another time we have

$$H^{(n+2)-p,p}_{\text{prim}}(X^2_d) \cong R_{(p+1)d-s-2t} \cong R_{pd-s} \cong H^{n-p+1,p-1}_{\text{prim}}(X^0_d),$$

where we set $H^{r,s}_{\text{prim}}(X) = 0$ if $r < 0$ or $s < 0$. On the other end the same yields for X^1_d with

$$H^{(n+3)-p,p}_{\text{prim}}(X^3_d) \cong R_{(p+1)d-s-3t} \cong R_{pd-s-t} \cong H^{(n+1)-p+1,p-1}_{\text{prim}}(X^1_d).$$

Every time we perform a double extension we end up in the same graded components of the Jacobian ring, just with shift in the Hodge theory, exactly as concluded above. □

Remark 2.2.4. Notice in particolar that, since the degree of any member of the tower is constant, we will always have $H^1(T_{X^k_d})_{\text{proj}} \cong R_d$. We recall that the distinction between $H^1(T_{X^k_d})_{\text{proj}}$ and $H^1(T_{X^k_d})$ holds only in dimension ≤ 2.

Remark 2.2.5. As showed for example in [105], the Jacobian ring of a variety actually determines its (IVHS). Therefore what we have is indeed an isomorphism of IVHS

$$\phi : H^n(X^0_d) \xrightarrow{\sim} H^{n+2k}(X^{2k}_d)[-2k].$$

2.2.1 Periodicity in Hodge theory and Torelli problem

Let us now focus on the case of Fano threefolds of index > 1, and pick any $X_d \subset \mathbb{P}(a_0, \dots, a_n)$. To be in a tower one of the weights a_i has to be $a_i = d/2$. If not, one can start running the game directly from $X_d = X^0_d$, adjoining a variable of half the weight of the degree. Of course if $d \equiv 1 \pmod 2$ there is no hope of building any tower.
Looking at the table of Okada, it turns out that 30 families of Fano threefolds of index > 1 lies in a tower. We are interested in the towers whose even member has an Hodge structure of K3-type. Note that to be Fano of K3 type, they have to satisfy $t = \iota_X$,

where t is the covering variable as above. Among all, 4 of them are of K3-type, and we will focus our attention on these in particular. They corresponds to the families

1. No. 97, $X_4 \subset \mathbb{P}(1, 1, 1, 1, 2)$;
2. No. 107, $X_6 \subset \mathbb{P}(1, 1, 2, 2, 3)$;
3. No. 116, $X_{10} \subset \mathbb{P}(1, 2, 3, 4, 5)$;
4. No. 122, $X_{14} \subset \mathbb{P}(2, 3, 4, 5, 7)$.

Tower on quartic double solid

The first example that we are going to consider is $X_4 \subset \mathbb{P}(1^4, 2)$ with coordinates $x_0, \dots, x_3, y_1$, already famous in literature as the quartic double solid. It is a double cover of $\mathbb{P}^3$ ramified on a quartic (K3) surface. Note that the threefold itself is smooth (and not only quasi-smooth): in fact, the generic member of the family will have $y_1^2 + \dots$, and therefore will avoid the coordinate point $P_4 = [0, 0, 0, 0, 1]$.
An infinitesimal Torelli theorem for the quartic double solid was already estabilished by Clemens ([39]). Moreover, since it shares numerical coincidences (in particular, the dimension of the intermediate Jacobian) with the Gushel-Mukai Fano threefold of index 10, Y_{10}, Tyurin conjectured the existence of a birational isomorphism between Y_{10} and X_4. This was just recently disproved by Debarre, Iliev, Manivel in [83].
If we consider the double cover X_4^2 of $\mathbb{P}(1, 1, 1, 1, 2, 2)$ branched on the quartic doube solid (we use this notation because we consider the K3 surface as the base of the tower) we will have that the resulting variety will be quasi-smooth, acquiring $2 \times \frac{1}{2}(1, 1, 1, 1)$ points on the intersection with the weighted $\mathbb{P}(2, 2)$. If we compute the Hilbert-Poincaré series of the Jacobian ring of any member of the tower, this will be

$$\mathrm{HP}(R) = 1 + 4t + 10t^2 + 16t^3 + 19t^4 + 16t^5 + 10t^6 + 4t^7 + t^8.$$

The odd Hodge structure will be concentrated in degrees 2 and 6 (with $R_2 \cong R_6$ both 10 dimensional), while the even Hodge structure will be in degrees $0, 4, 0$. It follows that for example the Fano threefold will have as Hodge Diamond

$$\begin{array}{ccccccc} 0 & & 10 & & 10 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

and the Fano fourfold will have

$$\begin{array}{ccccccccc} 0 & & 1 & & 20 & & 1 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

and the same for every even (and odd) dimension. Note that even in higher dimension the period map will depend only by the K3 structure (see for example [47]): in particular any member of the tower will satisfy the Torelli property. A further result regards the rationality property: while it is known that a smooth quartic double solid is irrational , the same does not hold in higher dimensions. We prove in fact

Proposition 2.2.6. *Let $X_4^j \subset \mathbb{P}(1^4, 2^j)$ a quasi-smooth member of the tower of quartic double space with dim $X_4^j \geq 4$. Then X_4^j is rational.*

Proof. Any quasi-smooth $X_4^j \subset \mathbb{P}(1^4, 2^j)$ is given up to $\mathbb{P}(1^4, 2^j)$-automorphism by an equation like:

$$X_4^j := (f(x_0, x_1, x_2, x_3, y_1, \dots, y_{j-2}) + y_{j-1}y_j = 0)$$

where $f(x_0, x_1, x_2, x_3, y_1, \dots, y_{j-2})$ is a quasi-smooth polynomial of degree 4. Now let us consider the open sub-scheme $U_j \hookrightarrow \mathbb{P}(1^4, 2^j)$ given by $y_j = 1$. We set $V_j := U_j \cap X_4^j$. By definition there exists a birational morphism between the affine variety $\mathbb{A}^{j-2}_{x_0,x_1,x_2,x_3,y_1,\dots,y_{j-2}} \dashrightarrow V_j$ given by

$$\phi \colon (x_0, x_1, x_2, x_3, y_1, \dots, y_{j-2}) \mapsto (x_0, x_1, x_2, x_3, y_1, \dots, y_{j-2}, f(x_0, x_1, x_2, x_3, y_1, \dots, y_{j-2})).$$

□

A second example of K3 type

The other example of K3 type that we want to investigate is the family no.122, $X_{14} \subset \mathbb{P}(2, 3, 4, 5, 7)$. This is particularly interesting, since is the only Fano of K3 type that represents as well a counterexample to the infinitesimal Torelli problem. We have already investigate the Hilbert-Poincaré series of the Jacobian ring, this being

$$\mathrm{HP}(R) = 1 + t^2 + t^3 + 2t^4 + 2t^5 + 3t^6 + 3t^7 + 5t^8 + 4t^9 + 6t^{10}+$$

$$+5t^{11} + 7t^{12} + 6t^{13} + 7t^{14} + 6t^{15} + 7t^{16} + 5t^{17}+$$

$$+6t^{18}+4t^{19}+5t^{20}+3t^{21}+3t^{22}+2t^{23}+2t^{24}+t^{25}+t^{26}+t^{28}.$$

Here we see that the Hodge theory in even dimension is concentrated in degree $1, 14$, while in odd dimension in degree $7, 21$. It follows that for example the Fano threefold will have as Hodge Diamond

$$\begin{array}{ccccccc} 0 & & 3 & & 3 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

and the Fano fourfold will have

$$\begin{array}{ccccccccc} 0 & & 1 & & 8 & & 1 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

with 8 being exactly the Picard rank of the K3 surface that is the step 0 of the tower. A curious behaviour appears here: one has in fact

Theorem 2.2.7. *Any odd dimensional member* $X_{14}^{2k+1} \subset \mathbb{P}(2, 3, 4, 5, 7^{2k+1})$ *will be of Anti-Torelli type, while any even dimensional member* $X_{14}^{2k} \subset \mathbb{P}(2, 3, 4, 5, 7^{2k})$ *will be of Torelli type. In particular we have an infinite chain of examples and counterexamples for the Torelli problem, with alternate dimensions.*

Proof. The failure in any odd dimension follows from the same reasons explained in 2.1.10. On the other hand in even dimension one has to check the (trivial) injectivity of the map

$$\mathrm{d}\,\wp : R_{14} \longrightarrow \mathrm{Hom}(R_0, R_{14} \oplus \mathbb{C}) \cong \mathrm{Hom}(\mathbb{C}, R_{14} \oplus \mathbb{C}).$$

The result follows immediately then. $\square$

The same construction can be performed starting from most of the $\mathbb{Q}$-Fano threefold hypersurfaces. We do not include here the complete list of towers coming out from the Fano threefolds. We refer indeed to [59] for both the complete lists.

Chapter 3

Hodge Theory and deformations of affine cones

We now present the first generalisation of Griffiths residue calculus. We show how to canonically associate to a smooth projective variety X (with some hypotheses of algebraic regularity, but no hypotheses on the codimension) a module $T^1_{A_X}$ parametrising the first order, infinitesimal deformations of its affine cone A_X. We show how from this module and its generalisation is possible to reconstruct the Hodge groups of X. In particular this technique, restricted to the hypersurface case, provide an alternative proof of Griffiths result.

3.1 What is the module T^1?

From now on we will focus on the study of the deformation modules attached to an affine cone over a smooth projective variety. Throughout the rest of the chapter fix X to be a smooth projectively normal variety and call A_X the affine cone over it, so that $A_X = \mathrm{Spec}(\mathfrak{a}_X)$, where

$$\mathfrak{a}_X = \bigoplus_k H^0(X, \mathcal{O}_X(k)).$$

We will also suppose everywhere that the projective embedding of X is subcanonical, so that $\omega_X \cong \mathcal{O}_X(m)$ for some integer m. Key references for most of the considerations below are [108, 117]. We stress that their results hold in the more general case of an affine isolated singularity, but here we only consider the case of affine cones over smooth projective varieties.

Consider the T^1-deformation module

$$T^1_{A_X} := \mathrm{Ext}^1_{\mathcal{O}_{A_X}}(\mathbb{L}_{A_X}, \mathcal{O}_{A_X}),$$

which measures the first-order deformations of the affine cone A_X: in particular notice that it is equipped with a natural grading, induced by the fact that $\mathfrak{a}_X$ is a graded algebra itself. This definition above in valid in way more generality, and in particular $\mathbb{L}_{A_X}$ stands for the full cotangent complex associated to A_X. However, with the above hypotheses on X and A_X, we can take as definition of $T^1_{A_X}$ the much more simple

$$T^1_{A_X} := \mathrm{Ext}^1_{\mathcal{O}_{A_X}}(\Omega^1_{A_X}, \mathcal{O}_{A_X}).$$

The cone A_X may deform in several ways and each graded component of $T^1_{A_X}$ roughly speaking represents the degree of the polynomial we are adding in order to deform. In the case X is a degree d hypersurface with defining polynomial f the graded module $T^1_{A_X}$ coincides, up to a shift of $-d$, with the Jacobian ring R_f, i.e. we have

$$T^1_{A_X}[-d] \cong R_f$$

as graded modules.
Now, not all deformations of A_X lead to another affine cone over a projective variety. The easiest example is the case of $xy = 0$ in $\mathbb{C}^2$ that is the cone over the points $[1,0], [0,1] \in \mathbb{P}^1$. In this case, up to isomorphism the only possible first-order deformation of the cone is given by $xy + \varepsilon = 0$, and any element of this family is not a cone over a projective variety except for $\varepsilon = 0$. In fact this is straightforward to verify: just notice that the Jacobian ring is

$$(R_f)_m \cong (\mathbb{C}[x,y]/(x,y))_m \cong \begin{cases} \mathbb{C} \text{ if } m = 0 \\ 0 \text{ if } m \neq 0 \end{cases},$$

hence $(T^1_{A_X})_0 = (R_f)_2 = 0$. A very natural question arises: what are the deformations of A_X that lead to family of affine cones over a projective variety? The naive answer is that the polynomials that we add in order to deform must be homogeneous of the same degree as the (homogeneous) equations of A_X. Luckily, this is also the correct one.
So far, all of what we said is very classical: under the above interpretation, the degree 0 piece of the deformation module of the affine cone over the projective variety X represents the embedded first-order deformations of X inside $\mathbb{P}^N$. Somehow more precisely, we have

an exact sequence

$$\ldots \to H^1(X, \mathcal{O}_X) \to (T^1_{A_X})_0 \to H^1(X, T_X) \to H^2(X, \mathcal{O}_X) \to \ldots$$

Notice that if the two side terms are both zero, we have an isomorphism between $(T^1_{A_X})_0$ and $H^1(X, T_X)$, which allows us to identify $(T^1_{A_X})_0$ with all infinitesimal deformations of X. This is for example the case of a smooth Calabi-Yau of dimension ≥ 3; by definition we have $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. This agrees with the standard fact that all Calabi-Yau from dimension 3 onwards are projective, while for example in the K3 case we have a 19-dimensional algebraic family inside a 20-dimensional deformation space, recorded by the fact $H^2(X, \mathcal{O}_X) \cong H^0(X, \omega_X) \cong \mathbb{C}$. In general when either $H^{0,1}(X)$ or $H^{0,2}(X)$ are non zero there is generally a difference between degree 0 embedded deformations and non-embedded ones.

3.1.1 T^1 and Hodge theory

Now we want to explore deeper the relation between $T^1_{A_X}$, the (quasi) smooth projective variety X, and the punctured cone $U_X := A_X \setminus \{0\}$.

Lemma 3.1.1. *For every $k \in \mathbb{Z}$, the relative tangent sheaf exact sequence*

$$0 \to T_{U_X/X} \to T_{U_X} \xrightarrow{\mathrm{d}\pi} \pi^*(T_X) \to 0$$

induces a long exact sequence

$$\ldots \to H^1(X, \mathcal{O}_X(k)) \to H^1(U_X, \Theta_{U_X})_k \to H^1(X, T_X(k)) \xrightarrow{\lambda} H^2(X, \mathcal{O}_X(k)) \to \ldots.$$

where the maps λ are the Lefschetz operators (that is, the cupping with $c_1(\mathcal{O}_X(1))$.

Proof. The Euler vector field gives a trivialisation $T_{U_X/X} \cong \mathcal{O}_{U_X}$, see [6]. We therefore get the short exact sequence

$$0 \to \mathcal{O}_{U_X} \to T_{U_X} \to \pi^*(T_X) \to 0,$$

and so, passing to cohomology, the long exact sequence

$$\ldots \to \bigoplus_{k \in \mathbb{Z}} H^1(X, \mathcal{O}_X(k)) \xrightarrow{\lambda} \bigoplus_{k \in \mathbb{Z}} H^1(U_X, \Theta_{U_X})_k \to$$

$$\to \bigoplus_{k\in\mathbb{Z}} H^1(X, T_X(k)) \to \bigoplus_{k\in\mathbb{Z}} H^2(X, \mathcal{O}_X(k)) \to \dots$$

where the grading on $H^1(U_X, T_{U_X})$ is induced by the $\mathbb{C}^*$-action, and the connecting homomorphism λ is the cup product with the extension class

$$\Lambda := [0 \to \mathcal{O}_{U_X} \to T_{U_X} \to \pi^* T_X \to 0]\,,$$

which is an element in

$$\mathrm{Ext}^1_{U_X}(\pi^* T_X, \mathcal{O}_{U_X}) \cong H^1(U_X, \pi^*\Omega^1_X) \cong \bigoplus_s H^1(X, \Omega^1_X(s)),$$

see [108, Lemma 1, page 158] and [109]. Notice that the map λ is not a priori a morphism of graded modules: we should expect it to have several homogeneous components

$$\lambda_s : H^i(X, T_X(k)) \longrightarrow H^{i+1}(X, \mathcal{O}_X(k+s)),$$

which are identified with cohomology classes in $H^1(X, \Omega^1(s))$. So our next step consists in showing that actually λ reduces to its degree zero component λ_0, i.e., that Λ consists into a single cohomology class in $H^1(X, \Omega^1)$. To see this, recall from [5] (see [74] for a more modern treatment) that given a line bundle L on a smooth complex manifold X, if we denote by $L^\bullet$ the total space of the dual bundle L^* with the zero section removed, then we have a canonical short exact sequence of sheaves of $\mathcal{O}_{L^\bullet}$-modules

$$0 \to \pi^*\Omega^1_X \to \Omega^1_{L^\bullet} \to \mathcal{O}_{L^\bullet} \to 0.$$

Pushing forward to X we get for every $k \in \mathbb{Z}$ a short exact sequence

$$0 \to \Omega^1_X(k) \to \mathcal{L}_k \to L^{\otimes k} \to 0,$$

where $\mathcal{L}_k$ denotes the degree k component of $\pi_*\Omega^1_{L^\bullet}$. In particular, if $L = \mathcal{O}_X(1)$, so that $L^\bullet \cong U_X$, we get the short exact sequences

$$0 \to \Omega^1_X(k) \to (\pi_*\Omega^1_{U_X})_k \to \mathcal{O}_X(k) \to 0,$$

and the projection formula together with the isomorphisms $\pi^*\mathcal{O}_X \cong \mathcal{O}_{U_X} \cong \pi^*\mathcal{O}_X(k)$, shows that these are indeed all obtained from the single short exact sequence

$$0 \to \Omega^1_X \to (\pi_*\Omega^1_{U_X})_0 \to \mathcal{O}_X \to 0$$

by tensoring it by $\mathcal{O}_X(k)$. This implies that the extension class $[0 \to \Omega^1_X(k) \to (\pi_*\Omega^1_{U_X})_k \to \mathcal{O}_X(k) \to 0]$ is actually independent of k, and so the total extension class $[0 \to \pi^*\Omega^1_X \to \Omega^1_{U_X} \to \mathcal{O}_{U_X} \to 0]$ reduces to the extension class $[0 \to \Omega^1_X \to (\pi_*\Omega^1_{U_X})_0 \to \mathcal{O}_X \to 0]$, which is an element in $\mathrm{Ext}^1_X(T_X, \mathcal{O}_X) \cong H^1(X, \Omega^1_X)$, see [21]. Since the short exact sequence $0 \to \pi^*\Omega^1_X \to \Omega^1_{U_X} \to \mathcal{O}_{U_X} \to 0$ is the dual of the short exact sequence $0 \to \mathcal{O}_{U_X} \to T_{U_X} \to \pi^*(T_X) \to 0$ we finally see that the connecting homomorphism λ is indeed of degree zero and is given by the cup product with a distinguished element Λ in $H^1(X, \Omega^1_X)$. Since λ is a degree zero operator, it preserves the gradings, and so for every degree k we have a long exact sequence

$$\ldots \to H^1(X, \mathcal{O}_X(k)) \xrightarrow{\lambda} H^1(U_X, T_{U_X})_k \to H^1(X, T_X(k)) \to H^2(X, \mathcal{O}_X(k)) \to \ldots$$

To conclude we have to identify Λ with the class of an hyperplane. Again, we refer to [5], where it is shown that, for a general line bundle L, the extension class $[0 \to \Omega^1_X \to \mathcal{L}_0 \to \mathcal{O}_X \to 0]$ is the first Chern class $c_1(L)$. So, for $L = \mathcal{O}_X(1)$ we find that the extension class is $c_1(\mathcal{O}_X(1))$, as desired. □

Corollary 3.1.2. *Let X be a smooth subcanonical projectively normal variety of dimension n, and let $m \in \mathbb{Z}$ be the integer such that $\omega_X \cong \mathcal{O}_X(m)$. Then the relative tangent sheaf exact sequence*

$$0 \to T_{U_X/X} \to T_{U_X} \xrightarrow{\mathrm{d}\pi} \pi^*(T_X) \to 0$$

induces a long exact sequence

$$\ldots \to H^{n-1,0}(X) \xrightarrow{\lambda} H^{n,1}(X) \to H^1(U_X, T_{U_X})_m \to H^{n-1,1}(X) \xrightarrow{\lambda} H^{n,2}(X) \to \ldots$$

where the maps λ are the Lefschetz operators.

Proof. Since $\omega_X \cong \mathcal{O}_X(m)$, Serre duality gives canonical isomorphisms $H^i(X, T_X(m)) \simeq H^{n-1,i}(X)$ and $H^i(X, \mathcal{O}_X(m)) \simeq H^{n,i}(X)$. The result then follows from Lemma 3.1.1 for $k = m$. □

We are now ready to prove the main result of this section.

Theorem 3.1.3. *Let X be a smooth subcanonical projectively normal variety of dimension n, and let $m \in \mathbb{Z}$ be the integer such that $\omega_X \cong \mathcal{O}_X(m)$. There is a natural isomorphism*

$$H^1(U, T_{U_X})_m \cong H^{n-1,1}_{\mathrm{prim}}(X).$$

Proof. Consider the long exact sequence

$$\dots \to H^{n-1,0}(X) \overset{\lambda_{n-1,0}}{\to} H^{n,1}(X) \to H^1(U_X, T_{U_X})_m \to H^{n-1,1}(X) \overset{\lambda_{n-1,1}}{\to} H^{n,2}(X) \to \dots$$

from Corollary 3.1.2. It induces the short exact sequence

$$0 \to \mathrm{coKer}(\lambda_{n-1,0}) \to H^1(U_X, T_{U_X})_m \to \mathrm{Ker}(\lambda_{n-1,1}) \to 0.$$

Now notice that, by definition, $\mathrm{Ker}(\lambda_{n-1,1}) = H^{n-1,1}(X)_{\mathrm{prim}}$, while $\mathrm{coKer}(\lambda_{n-1,0})$ is zero since, $\lambda_{n-1,0}$ is an isomorphism by Hard Lefschetz. $\square$

What we have to do now is connect the previous result to the $T^1_{A_X}$, the module of first-order deformations of the affine cone of X.

Theorem 3.1.4. *Let X be a smooth subcanonical projectively normal variety of dimension $n > 1$, and let $m \in \mathbb{Z}$ be the integer such that $\omega_X \cong \mathcal{O}_X(m)$. If $H^1(X, \mathcal{O}_X(k)) = 0$ for every $k \in \mathbb{Z}$, then we have*

$$(T^1_{A_X})_m \cong H^{n-1,1}_{\mathrm{prim}}(X)$$

Proof. From [108] we have that $T^1_{A_X}$ fits into the exact sequence

$$0 \to T^1_{A_X} \to H^1(U_X, T_{U_X}) \to H^1(U_X, (T_{\mathbb{C}^{N+1}})|_{U_X}) \cong H^1(U_X, \mathcal{O}_{U_X})^{N+1}.$$

Since $H^1(U_X, \mathcal{O}_{U_X}) \cong \bigoplus_k H^1(X, \mathcal{O}_X(k))$, we see that if $H^1(X, \mathcal{O}_X(k)) = 0$ for every $k \in \mathbb{Z}$, then $T^1_{A_X} \cong H^1(U_X, T_{U_X})$. The conclusion then follows from Theorem 3.1.3. $\square$

Corollary 3.1.5. *Under the same hypothesis of the theorem above, we have that in general the degree k component of the $T^1_{A_X}$ is given by*

$$(T^1_{A_X})_k \cong \mathrm{Ker}(\lambda : H^1(X, \Omega^{n-1}(k-m)) \to H^2(X, \omega_X(k-m)).$$

Remark 3.1.6. What kind of varieties satisfies the condition $H^1(X, \mathcal{O}_X(k)) = 0$ for every $k \in \mathbb{Z}$ that appears in Theorem 3.1.4 above? By Kodaira vanishing one sees that all smooth Fano manifolds and simply connected projective Calabi-Yau manifolds satisfy this condition. Also, every arithmetically Cohen-Macaulay projective variety (and so, in particular projective spaces and their products, projective complete intersections, Grassmann manifolds and Schubert subvarieties, flag manifolds and generalised flag manifolds) of dimension at least 2 satisfy it. Notice that, if $\dim X \geq 2$, the vanishing condition

condition $H^1(X, \mathcal{O}_X(k)) = 0$ for every $k \in \mathbb{Z}$ is actually equivalent to the condition $\mathrm{depth}_0 A_X \geq 3$. Namely, we can identify $H^1(U_X, \mathcal{O}_{U_X})$ with $H^2_{\mathfrak{m}}(A_X, \mathcal{O}_{A_X})$, the second local cohomology group of A_X at the maximal (irrelevant) ideal, and the vanishing of this is by definition the same request as $\mathrm{depth}_0 A_X \geq 3$.

For $\dim X \geq 2$ the simplest example of projective manifolds for which $H^1(X, \mathcal{O}_X(k))$ does not vanish for every k given by Abelian varieties: for them, theorem 3.1.3 still holds, but the problem of determining the image of T^1_A inside $H^1(U_X, T_{U_X})$ remains open.

3.1.2 Obstructions and automorphisms

Now we look at the obstruction theory of the cone A_X. Infinitesimal obstructions to deformations of A_X live inside

$$T^2_{A_X} := \mathrm{Ext}^2_{\mathcal{O}_{A_X}}(\Omega^1_{A_X}, \mathcal{O}_{A_X}).$$

Let us stick to the case of $\mathrm{depth}_0 A_X \geq 3$ and $\dim X \geq 2$, so that $H^1(X, \mathcal{O}_X(k)) = 0$ for any k as in the previous section. Following [108], we can identify $T^2_{A_X}$ with $H^1(U_X, N_{U_X})$, where N_{U_X} is the normal bundle of U_X in $\mathbb{C}^{N+1}$. From the defining exact sequence

$$0 \to T_{U_X} \to T_{\mathbb{C}^{N+1}}|_{U_X} \to N_{U_X} \to 0$$

for N_{U_X} we obtain the long exact sequence

$$\ldots \to 0 \to H^1(U_X, N_{U_X}) \to H^2(U_X, T_{U_X}) \to \left(\bigoplus_k H^2(X, \mathcal{O}_X(k))\right)^{N+1} \to \ldots \tag{3.1}$$

in cohomology, so that if for all k $H^2(X, \mathcal{O}_X(k)) = 0$ we have an isomorphism

$$T^2_{A_X} \cong H^2(U_X, T_{U_X}).$$

Notice that the condition $H^i(X, \mathcal{O}_X(k)) = 0$ for any k in $\mathbb{Z}$ and for $i = 1, 2$ is in particular satisfied by every arithmetically Cohen-Macaulay variety of dimension at least 3. From Lemma 3.1.1 we have the exact sequences

$$\ldots \to 0 \to (T^2_{A_X})_k \to H^2(X, T_X(k)) \xrightarrow{\lambda} H^3(X, \mathcal{O}_X(k)) \to \ldots$$

where the maps λ are the Lefschetz operators. Let us restrict to the cases $k = 0$ and $k = m$, where m is the integer such that $\omega_X \cong \mathcal{O}(m)$. For $k = 0$ we find the exact sequence

$$\ldots \to 0 \to (T^2_{A_X})_0 \to H^2(X,T_X) \xrightarrow{\lambda} H^3(X,\mathcal{O}_X) \to \ldots$$

which identifies $(T^2_A)_0$ with a subspace of $H^2(X,T_X)$, which is the space containing the obstruction to (non-immersed) deformations of X. If moreover $\lambda\colon H^2(X,T_X) \to H^3(X,\mathcal{O}_X)$ is the zero map (as is the case, e.g., if $H^3(X,\mathcal{O}_X)$ vanishes), then we have an isomorphism $(T^2_A)_0 \cong H^2(X,T_X)$. If instead we look at the $k=m$ case, then by Corollary 3.1.2 we have the long exact sequence

$$\ldots \to H^{n-1,1}(X) \xrightarrow{\lambda} H^{n,2}(X) \to (T^2_A)_m \to H^{n-1,2}(X) \xrightarrow{\lambda} H^{n,3}(X) \to \ldots$$

and so we get the following

Theorem 3.1.7. *Let X be a smooth subcanonical projectively normal variety of dimension n, and let $m \in \mathbb{Z}$ be the integer such that $\omega_X \cong \mathcal{O}_X(m)$. If $H^i(X,\mathcal{O}_X(k)) = 0$ for every $k \in \mathbb{Z}$, and for $i = 1,2$ then we have a natural isomorphism*

$$(T^2_A)_m \cong H^{n-2,1}_{\text{prim}}(X).$$

Proof. By the above discussion, we have a natural short exact sequence

$$0 \to \operatorname{coKer}(\lambda_{n-1,1}) \to (T^2_A)_m \to \operatorname{Ker}(\lambda_{n-1,2}) \to 0.$$

By Hard Lefschetz, $\operatorname{coKer}(\lambda_{n-1,1}) = 0$, and

$$\operatorname{Ker}(\lambda_{n-1,2}) = \lambda_{n-2,1}\operatorname{Ker}(\lambda_{n-1,2}\lambda_{n-2,1}) \cong H^{n-2,1}_{\text{prim}}(X).$$

□

A similar result holds for the $T^0_{A_X}$, the module that parametrize the infinitesimal automorphism of the affine cone A_X. If $\text{depth}_0 A_X \geq 2$ (which is satisfied, e.g., if X is normal), we have $T^0_{A_X} \cong H^0(U_X, T_{U_X})$ and so Corollary 3.1.2 gives the long exact sequence

$$0 \to H^{n,0}(X) \to (T^0_A)_m \to H^{n-1,0}(X) \xrightarrow{\lambda_{n-1,0}} H^{n,1}(X) \to \ldots$$

and so the short exact one

$$0 \to H^{n,0}(X) \to (T^0_A)_m \to \operatorname{coKer}(\lambda_{n-1,0}) \to 0.$$

By Hard Lefschetz, $\lambda_{n-1,0}$ is an isomorphism and so $\operatorname{coKer}(\lambda_{n-1,0}) = 0$, while $H^{n,0}(X) =$

$H^{n,0}_{\text{prim}}(X)$. Thus we have

Theorem 3.1.8. *Let X of dimension n be smooth, projective, with $\omega_X \cong \mathcal{O}_X(m)$. Then we have an isomorphism*

$$(T^0_A)_m \cong H^{n,0}_{\text{prim}}(X).$$

3.1.3 A SINGULAR appendix: how to compute Hodge numbers using the T^i

One of the many applications of our theorems on the $T^1_{A_X}$ and the $T^2_{A_X}$ is a concrete tool to compute part of the Hodge structure of a smooth projective variety. We recall from the previous section that, under appropriate hypothesis on depth at the vertex, we can identify

$$(T^1_{A_X})_m \cong H^{n-1,1}_{\text{prim}}(X)$$

$$(T^2_{A_X})_m \cong H^{n-2,1}_{\text{prim}}(X),$$

where as usual $\omega_X \cong \mathcal{O}_X(m)$. The key is that both T^1 and T^2 are easily computable, especially using computer algebra languages such as SINGULAR (see [51]), already endowed with efficient built-in tools. Suppose we start from $X \subset \mathbb{P}^N$ a smooth projective variety, with $X = V(I)$, where $I = (f_1, \dots, f_m)$. Then the instruction

```
module T_1= T_1(I);
module T_2=T_2(I);
hilb(T_1,2);
hilb(T_2,2);
```

computes the dimension of both $T^1_{A_X}[-d]$ and $T^2_{A_X}[-d]$, where $d = \max\{\deg(f_i)\}$ and then lists the dimensions of the various graded components. Let us pursue a couple of nontrivial example in detail:

A (Gushel-Mukai) Fano Threefold of Degree 10 and Coindex 1

Consider the case of $X = \text{Gr}(2,5) \cap H_1 \cap H_2 \cap Q$ a threefold complete intersection in the Grassmannian of 2-planes in $\mathbb{C}^5$ given by two hyperplane sections and one quadric, considered for example in [88] and [47] for its connections with hyperKähler geometry. Since the canonical class of the Grassmannian is $\omega_{\text{Gr}(2,5)} \cong \mathcal{O}_{\text{Gr}(2,5)}(-5)$, by adjunction X is a Fano of index 1, that is $\omega_X = \mathcal{O}_X(-1)$. By Kodaira vanishing we have $h^{i,0}(X) = 0$ for $i = 1,2,3$, and by Lefschetz hyperplane theorem we have $H^{1,1}(X) = H^{2,2}(X) \cong \mathbb{C}$.

Thus the only Hodge piece missing is $H^{2,1}(X) = H^{2,1}_{\mathrm{prim}}(X)$, and by our theorem we have

$$H^{2,1}_{\mathrm{prim}}(X) \cong (T^1_{A_X})_{-1}.$$

Let us produce a code in SINGULAR:

```
ring r=97, (y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9), ds;
ideal I =y_2*y_4-y_1*y_5+y_0*y_7, y_3*y_4-y_1*y_6+y_0*y_8,
y_3*y_5-y_2*y_6+y_0*y_9, y_3*y_7-y_2*y_8+y_1*y_9, y_6*y_7-y_5*y_8+y_4*y_9,
sparsepoly(1,1,0,10),sparsepoly(1,1,0,10),sparsepoly(2,2,0,10);
module T= T_1(I);
hilb(T,2);
```

the first 5 equations in the ideal are nothing but the Plücker relations of $Gr(2,5)$ embedded in $\mathbb{P}^9$, with three generic hyperplane sections and one quadratic equation (actually SINGULAR can check if those four equations form a regular sequence). The dimension of the graded component we get in return is

```
1,10,22,11,1
```

where we have to look at the component of degree 1, since $(T^1_{A_X})_{-1} = (T^1_{A_X})[-2]_1$, and this is 10. If needed, we can ask for an explicit monomial basis for $H^{2,1}(X)$, using the command

```
kbase(T,1);
```

Note that $(T^1_{A_X})_0$ is 22-dimensional: this agrees (and the same for the Hodge numbers) with the computation considered in [48, 47]. In particular its lower Hodge diamond is

$$\begin{array}{ccccccc} 0 & & 10 & & 10 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

A (quadratic section of) Pfaffian-Calabi Yau Threefold

Consider now the case of a Pfaffian-Calabi Yau Threefold, as in the work of [72]. We define

$$P = G \cap Q_1 \cap Q_2 \cap H,$$

where $G = Gr(2,5)$ as before, Q_1 and Q_2 are quadrics, H is an hyperplane. By a computation analogous to the previous example it is easy to see that P is a Calabi-Yau threefold: the dimension of the graded component of the T^1 of its affine cone are

```
2,19,61,101,82, 29, 3
```

In particular we find the lower Hodge diamond

$$\begin{array}{ccccccc} 1 & & 61 & & 61 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

A weighted example

As said before, the method we implemented works well also in the weighted projective space case. As an example, we look at the online database [24] where several thousands of families of quasi-smooth Fano threefolds are listed. In SINGULAR, we can deal with weighted projective space by specifying a weighted order on the monomial, namely using the command

```
wp(a_0,...,a_n)
```

where the a_i are the chosen weights. As an example, we pick $X_{6,7} \subset \mathbb{P}(1,1,2,2,3,5)$, corresponding to the entry 5839 in the database [24]. This is a codimension 2 Fano threefold of index 1, degree 7/10 and with a $3 \times \frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,2,3)$ as Basket. Computing the T^1 in the same exact way as before we get $(T^1_A)_{-1} = 39$, and we can then draw the lower Hodge diamond as

$$\begin{array}{ccccccc} 0 & & 39 & & 39 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

3.2 Deformations of derived categories and Hodge theory

3.2.1 A primer on noncommutative schemes and Hochschild structures

One natural way to generalise the notion of deformation of a $\mathbb{C}$-scheme Y is by considering derived deformations, i.e. deformations of Y as a derived scheme. Namely, derived deformations of Y are already encoded in the cotangent complex. On the other hand another interesting generalisation consists in deforming Y as a noncommutative scheme – whatever these structures could be.

The theory of noncommutative schemes, usually known as Noncommutative Algebraic Geometry, is very much a developing subject, whose fundamentals have not been completely settled yet; however the basic idea – which dates back to Grothendieck and has recently gone through a rapid development – consists of observing that the geometry of Y does not really depend on the scheme as a space, but rather on the derived category $D(\mathfrak{Coh}\,(Y))$ or better on its dg-enhancements or, more generally A_∞-enhancements. Therefore, a (non-necessarily commutative) scheme over $\mathbb{C}$ can be thought of the datum of an A_∞-category over $\mathbb{C}$: those which are quasi-equivalent to a dg-category of coherent sheaves over a classical scheme will encode the usual commutative schemes, whereas the others will be called noncommutative schemes. In particular, a noncommutative deformation of the commutative scheme Y will be a deformation of the dg-category category $D(\mathfrak{Coh}\,(Y))$ as an A_∞-category [14].

Exactly as the infinitesimal deformation theory of an associative algebra (or more generally an A_∞-algebra) is governed by its Hochschild cohomology, so happens for the infinitesimal deformation theory of A_∞-categories. In particular, the noncommutative deformations of a $\mathbb{C}$-scheme Y are governed by

$$\mathrm{HH}^*\,(Y) := \mathrm{HH}^*\,(D\mathfrak{Coh}(Y)) \cong \mathrm{Ext}^*_{Y\times Y}\,(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$$

where $\mathcal{O}_\Delta$ stands for the structure sheaf of the diagonal in $Y \times Y$, see [34].

There are some even more concrete interpretations of Hochschild cohomology: as a matter of fact if Y is a smooth quasiprojective variety the Hochschild—Kostant—Rosenberg Theorem (see [34]) establishes an isomorphism between $\mathrm{HH}^*(Y)$ and the (cohomology) algebra of polyvector fields on Y, namely

$$\mathrm{HH}^\bullet(Y) \cong \bigoplus_{p,q} H^q(Y, \bigwedge^p T_Y).$$

Notice that the algebra of polyvector fields $\bigoplus_{p,q} H^q(Y, \bigwedge^p T_Y)$ is known to be the tangent space at Y to the *extended moduli supermanifold of complex structures* (see [7]) and is also related to the derived moduli of non-commutative polarized schemes recently studied by Berhend and Noohi [12]. Finally, if Y is a projective Calabi-Yau, Serre duality gives a canonical isomorphism

$$\mathrm{HH}^\bullet(Y) \cong \mathrm{HH}_\bullet(Y)[n]$$

where the Hochschild homology of Y is identified the "vertical slices" of the Hodge diamond of Y:

$$\mathrm{HH}_\bullet(Y) = \bigoplus_k \mathrm{HH}_k(Y) = \bigoplus_k \left(\bigoplus_{p-q=k} H^{p,q}(Y) \right).$$

3.2.2 Hochschild cohomology of punctured affine cones

In the context of Hochschild structures, the results in Section 2.2 and Section 2.3 have beautiful generalisations. We start proving the following straightforward generalisation of Lemma 3.1.1.

Lemma 3.2.1. *For every $k \in \mathbb{Z}$,and for every $p \geq 0$, the relative tangent sheaf exact sequence*

$$0 \to T_{U_X/X} \to T_{U_X} \xrightarrow{\mathrm{d}\,\pi} \pi^*(T_X) \to 0$$

induces a long exact sequence

$$\ldots \to H^q(X, \bigwedge^{p-1} T_X(k)) \to H^q(U_X, \bigwedge^{p} T_{U_X})_k \to H^q(X, \bigwedge^{p} T_X(k)) \xrightarrow{\lambda} H^{q+1}(X, \bigwedge^{p-1} T_X(k)) \to \ldots$$

where the maps λ are the contractions with the hyperplane class in $H^1(X, \Omega^1_X)$, and where for $p = 0$ one is setting $\bigwedge^{-1} T_X = 0$.

Proof. The differential $\mathrm{d}\,\pi \colon T_{U_X} \to \pi^*(T_X)$ induces a short exact sequence of sheaves of $\mathcal{O}_{U_X}$-algebras

$$0 \to \ker(\mathrm{d}\,\pi) \to \bigwedge^{\bullet} T_{U_X} \xrightarrow{\mathrm{d}\,\pi} \pi^* \bigwedge^{\bullet} T_X \to 0, \tag{3.2}$$

which in every homogeneous degree p reads

$$0 \to \pi^* \bigwedge^{p-1} T_X \to \bigwedge^{p} T_{U_X} \to \pi^* \bigwedge^{p} T_X \to 0$$

since the leftmost term in the relative tangent sheaf exact sequence is a trivial line bundle (see, e.g., [67, Theorem 4.13]). Since (3.2) is a square zero extension, the connecting homomorphisms in the long exact sequence

$$\ldots \to H^q(U_X, \pi^* \bigwedge^{\bullet-1} T_X) \to H^q(U_X, \bigwedge^{\bullet} T_{U_X}) \to H^q(U_X, \pi^* \bigwedge^{\bullet} T_X) \xrightarrow{\lambda} H^{q+1}(U_X, \pi^* \bigwedge^{\bullet-1} T_X) \to \ldots$$

are given by the connecting homomorphism for the degree 1 sequence

$$\ldots \to H^q(U_X, \mathcal{O}_{U_X}) \to H^q(U_X, T_{U_X}) \to H^q(U_X, \pi^* T_X) \xrightarrow{\lambda} H^{q+1}(U_X, \mathcal{O}_{U_X}) \to \ldots$$

extended as a (graded) derivation. By Lemma 3.1.1 we know that this is given by the contraction with the hyperplane class seen as a degree zero element in $H^1(U_X, \pi^*\Omega^1_X)$. In particular, λ will be degree preserving, and so we get, for every p and every k the long exact sequence

$$\ldots \to H^q(X, \bigwedge^{p-1} T_X(k)) \to H^q(U_X, \bigwedge^{p} T_{U_X})_k \to H^q(X, \bigwedge^{p} T_X(k)) \xrightarrow{\lambda} H^{q+1}(X, \bigwedge^{p-1} T_X(k)) \to \ldots,$$

where λ is the contraction with the hyperplane class in $H^1(X, \Omega^1_X)$. $\square$

Assuming $\omega_X \cong \mathcal{O}_X(m)$, the nondegenerate pairings $\Omega^i_X \otimes \Omega^{n-i}_X \to \omega_X$ induce isomorphisms $\bigwedge^i T_X(m) \cong \Omega^{n-i}_X$. Under these isomorphisms, the contraction morphisms

$$H^q(X, \bigwedge^{p} T_X(m)) \xrightarrow{\lambda} H^{q+1}(X, \bigwedge^{p-1} T_X(m))$$

become the Lefschetz maps

$$H^{n-p,q}(X) \xrightarrow{\lambda} H^{n-p+1,q+1}(X).$$

Therefore we obtain the following.

Corollary 3.2.2. *Let X be a smooth subcanonical projectively normal variety of dimension n, and let $m \in \mathbb{Z}$ be the integer such that $\omega_X \cong \mathcal{O}_X(m)$. Then we have a long exact sequence*

$$\cdots \to H^{n-p,q-1}(X) \xrightarrow{\lambda_{n-p,q-1}} H^{n-p+1,q}(X) \to H^q(U_X, \bigwedge^{p} T_{U_X})_m \to H^{n-p,q}(X) \xrightarrow{\lambda_{n-p,q}} H^{n-p+1,q+1}(X) \to \ldots \tag{3.3}$$

where $\lambda_{i,j} \colon H^{i,j}(X) \to H^{i+1,j+1}(X)$ is the Lefschetz operator.

We can then prove the following result, expressing the Hochschild cohomology of the punctured cone U_X in terms of the primitive cohomology of X.

Theorem 3.2.3. *In the above assumptions we have a canonical isomorphism*

$$\mathrm{HH}^{p,q}(U_X)_m \cong H^{n-p+1,q}_{\mathrm{prim}}(X) \oplus H^{n-q,p}_{\mathrm{prim}}(X),$$

where, for each value of p, q, at most one of the two summands on the right is nonzero.

In particular,

$$\mathrm{HH}^{p,q}(U_X)_m \cong \begin{cases} H^{n-p+1,q}_{\text{prim}}(X) & \textit{if } p > q \\ \\ H^{n-q,p}_{\text{prim}}(X) & \textit{if } p \leq q. \end{cases}$$

Proof. The long exact sequence 3.3 induces the short ones

$$0 \to \mathrm{coKer}(\lambda_{n-p,q-1}) \to H^q(U, \bigwedge^p T_U)_m \to \mathrm{Ker}(\lambda_{n-p,q}) \to 0.$$

If $p \leq q$, then $\mathrm{coKer}(\lambda_{n-p,q-1}) = 0$ and $\mathrm{Ker}(\lambda_{n-p,q}) = \lambda^{-p+q} H^{n-q,p}_{\text{prim}}(X)$ by Hard Lefschetz, and so

$$H^q(U_X, \bigwedge^p T_{U_X})_m \cong H^{n-q,p}_{\text{prim}}(X)$$

in this case. Note that for $p = q$ this gives $H^p(U_X, \bigwedge^p T_{U_X})_m \cong H^{n-p,p}_{\text{prim}}(X)$.

If $p > q$, again by Hard Lefschetz we have $\mathrm{Ker}(\lambda_{n-p,q}) = 0$ and by definition $\mathrm{coKer}(\lambda_{n-p,q-1}) = H^{n-p+1,q}_{\text{prim}}(X)$, so that

$$H^q(U_X, \bigwedge^p T_{U_X})_m \cong H^{n-p+1,q}_{\text{prim}}(X)$$

in this case. By setting $H^{i,j}_{\text{prim}}(X) = 0$ if $i + j > n$, we can summarize the above results as

$$\mathrm{HH}^{p,q}(U_X)_m = H^q(U_X, \bigwedge^p T_{U_X})_m \cong H^{n-p+1,q}_{\text{prim}}(X) \oplus H^{n-q,p}_{\text{prim}}(X)$$

for any p, q.

□

The above Theorem 3.2.3 admits a nice rephrasing in terms of the derived deformation complex of A_X,

$$T^{p,q}_{A_X} := \mathrm{Ext}^q_{\mathcal{O}_{A_X}}(\wedge^p \mathbb{L}_{A_X}, \mathcal{O}_{A_X}).$$

Corollary 3.2.4. *Let X be a smooth subcanonical projectively normal variety of dimension n which is arithmetically Cohen-Macaulay, and let $m \in \mathbb{Z}$ be the integer such that $\omega_X \cong \mathcal{O}_X(m)$. Let A_X the affine cone of X. Then, for every $1 \leq p \leq n+1$ and $0 \leq q \leq n$, we have*

$$(T^{p,q}_{A_X})_m = \mathrm{Ext}^q_{\mathcal{O}_{A_X}}(\Omega^p_{A_X}, \mathcal{O}_{A_X})_m \cong \begin{cases} H^{n-p+1,q}_{\text{prim}}(X) & \textit{if } p > q \\ \\ H^{n-q,p}_{\text{prim}}(X) & \textit{if } p \leq q. \end{cases}$$

Proof. Since X is a smooth projective variety of dimension n, the affine cone A_X is smooth in codimension $n+1$ and so for $1 \le p \le n+1$ and $0 \le q \le n$ we have $T^{p,q}_{A_X} = \mathrm{Ext}^q_{\mathcal{O}_{A_X}}(\Omega^p_{A_X}, \mathcal{O}_{A_X})$, see, e.g., [60, Lemma 3.2]. Since X is also arithmetically Cohen-Macaulay, we have $\mathrm{depth}_0 A_X \ge n$ and this, following SGA 2 Exposè VI, [65], implies that the inclusion $U_X \hookrightarrow A_X$ induces an isomorphism $\mathrm{Ext}^q_{\mathcal{O}_{A_X}}(\Omega^p_{A_X}, \mathcal{O}_{A_X}) \cong \mathrm{Ext}^q_{\mathcal{O}_{U_X}}(\Omega^p_{U_X}, \mathcal{O}_{U_X})$. Finally, since U_X is smooth, we have $\mathrm{Ext}^q_{\mathcal{O}_{U_X}}(\Omega^p_{U_X}, \mathcal{O}_{U_X}) \cong \mathrm{HH}^{p,q}(U_X)$. $\square$

Although the above corollary is essentially a rephrasing of Theorem 3.2.3, it is important to stress that, when we consider the whole affine cone, the Ext modules becomes easy to compute using computer algebra software as SINGULAR or MACAULAY2 ([51], [62]). In particular it should be possible to write down a computer package - similar to the ne already existing for T^1 and T^2 - able to compute all of the Hodge numbers of a smooth projective arithmetically Cohen-Macaulay variety.

3.2.3 The case of a hypersurface

The results of the previous section lead to an interesting corollary in the case of a hypersurface: we can use them to recover Griffiths' isomorphism between the primitive cohomology of a smooth hypersurface X and a distinguished graded component of the Jacobian ring of a polynomial defining X. We start with some preliminary lemmata.

Lemma 3.2.5. *For X a smooth, projective hypersurface in $\mathbb{P}^{n+1}$ of degree d, we have a natural isomorphism*

$$H^{p-i}(U_X, \bigwedge^{p-i} T_{U_X})_{m+kd} \cong H^{p-i}(X, \bigwedge^{p-i} T_X(m+kd))$$

where $m = d-n-2$, for every $0 \le i \le p \le n$ and every $k \ge 1$.

Proof. Using the duality isomorphisms $\bigwedge^i T_X(m) \cong \Omega^{n-i}_X$, by Lemma 3.2.1 we have a short exact sequence

$$0 \to \mathrm{coker}\{\lambda_{n-p+i,p-i-1}(kd)\} \to H^{p-i}(U_X, \bigwedge^{p-i} T_{U_X})_{m+kd} \to \ker\{\lambda_{n-p+i,p-i}(kd)\} \to 0$$

where $\lambda_{i,j}(kd)$ is the Lefschetz morphism $H^j(X, \Omega^i_X(kd)) \to H^{j+1}(X, \Omega^{i+1}_X(kd))$. Since $H^{p-i}(X, \Omega^{n-p+i+1}_X(kd)) = 0$ and $H^{p-i+1}(X, \Omega^{n-p+i+1}_X(kd)) = 0$ by Kodaira vanishing, this reduces to

$$0 \to H^{p-i}(U_X, \bigwedge^{p-i} T_{U_X})_{m+kd} \to H^{p-i}(X, \Omega^{n-p+i}(kd)) \to 0$$

i.e., to

$$H^{p-i}(U_X, \bigwedge^{p-i} T_{U_X})_{m+kd} \cong H^{p-i}(X, \bigwedge^{p-i} T_X(m+kd))$$

$\square$

Corollary 3.2.6. *For every $p \geq 2$ we have natural isomorphisms*

$$H^1(U_X, T_{U_X})_{m+(p-1)d} \cong H^1(X, T_X(m+(p-1)d))$$

and

$$H^{p-1}(U_X, \bigwedge^{p-1} T_{U_X})_{m+d} \cong H^{p-1}(X, \bigwedge^{p-1} T_X(m+d))$$

Lemma 3.2.7. *Let X a smooth, projective hypersurface in $\mathbb{P}^{n+1}$ of degree d, with adjunction degree $m = d-n-2$. Then we have a natural isomorphism*

$$H^{p-i-1}(X, \bigwedge^{p-i-1} T_X(m+(k+1)d)) \cong H^{p-i}(X, \bigwedge^{p-i} T_X(m+kd))$$

for every $0 \leq i \leq p-2$, with $0 \leq p \leq n$

Proof. For any $0 \leq i \leq p \leq n$ consider the short exact sequence

$$0 \to \bigwedge^{p-i} T_X \to \bigwedge^{p-i} T_{\mathbb{P}^{n+1}}|_X \to \bigwedge^{p-i-1} T_X(d) \to 0$$

Using the duality isomorphisms $\bigwedge^i T_X(m) \cong \Omega_X^{n-i}$ and $\bigwedge^i T_{\mathbb{P}^{n+1}}|_X(m) \cong \Omega_{\mathbb{P}^{n+1}}^{n+1-i}|_X(d)$ we get to the sequence

$$\ldots \to H^{p-i-1}(X, \Omega_X^{n-p+i}(kd)) \to H^{p-i-1}(X, \Omega_{\mathbb{P}^{n+1}}^{n-p+i+1}((k+1)d)|_X) \to H^{p-i-1}(X, \Omega_X^{n-p+i+1}(k+1)d)) \to$$

$$\to H^{p-i}(X, \Omega_X^{n-p+i}(kd)) \to H^{p-i}(X, \Omega_{\mathbb{P}^{n+1}}^{n-p+i+1}((k+1)d)|_X) \to 0,$$

where the last zero comes from $H^{p-i}(X, \Omega_X^{n-p+i+1}((k+1)d)) = 0$ by Kodaira Vanishing. Now, consider the short exact sequence (see [15])

$$0 \to \Omega_{\mathbb{P}^{n+1}}^{n-p+i+1}(kd) \to \Omega_{\mathbb{P}^{n+1}}^{n-p+i+1}((k+1)d) \to \Omega_{\mathbb{P}^{n+1}}^{n-p+i+1}((k+1)d)|_X \to 0.$$

This induces the long exact sequence

$$\cdots \to H^{p-i}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}((k+1)d)) \to H^{p-i}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}((k+1)d)|_X) \to$$

$$\to H^{p-i+1}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}(kd)) \to H^{p-i+1}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}((k+1)d)) \to \cdots$$

By Kodaira vanishing we have $H^{p-i+1}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}((k+1)d)) = 0$. Also we have the long exact sequence

$$\cdots \to H^{p-i-1}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}((k+1)d)) \to H^{p-i-1}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}((k+1)d)|_X) \to$$

$$\to H^{p-i}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}(kd)) \to H^{p-i}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}((k+1)d)) \to \cdots$$

Since $i \neq p-1$, by Bott vanishing we also have

$$H^{p-i}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}((k+1)d)) = 0$$

and

$$H^{p-i-1}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}((k+1)d)) = 0.$$

Therefore we get

$$H^{p-i}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}((k+1)d)|_X) \cong H^{p-i+1}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}(kd)) = 0$$

and

$$H^{p-i-1}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}((k+1)d)|_X) \cong H^{p-i}(\mathbb{P}^{n+1}, \Omega^{n-p+i+1}_{\mathbb{P}^{n+1}}(kd)) = 0$$

where the rightmost zeroes come again by Bott vanishing, since $k \neq 0$ by hypothesis. So, we find

$$H^{p-i-1}(X, \Omega_X^{n-p+i+1}((k+1)d)) \cong H^{p-i}(X, \Omega_X^{n-p+i}(kd))$$

i.e.,

$$H^{p-i-1}(X, \bigwedge^{p-i-1} T_X(m+(k+1)d)) \cong H^{p-i}(X, \bigwedge^{p-i} T_X(m+kd))$$

$\square$

Corollary 3.2.8. *For any $2 \leq p \leq n$ and every $1 \leq i \leq p-1$ one has*

$$H^1(X, T_X(m+(p-1)d)) \cong H^{p-i}(X, \bigwedge^{p-i} T_X(m+id))$$

Proof. The proof is by descending induction on i. For $i = p-1$ there is nothing to prove. Assume the statement is true for $i+1$ and prove it for i. By the inductive hypothesis we have

$$H^1(X, T_X(m+(p-1)d)) \cong H^{p-i-1}(X, \bigwedge^{p-i-1} T_X(m+(i+1)d))$$

Since in the inductive step we have $i+1 \leq p-1$, and so we are in the hypothesis of the above Lemma. This gives

$$H^{p-i-1}(X, \bigwedge^{p-i-1} T_X(m+(i+1)d)) \cong H^{p-i}(X, \bigwedge^{p-i} T_X(m+id))$$

and we are done. □

Taking $i = 1$ in the above corollary we find

Corollary 3.2.9. *For any $2 \leq p \leq n$ one has a natural isomorphism*

$$H^1(X, T_X(m+(p-1)d)) \cong H^{p-1}(X, \bigwedge^{p-1} T_X(m+d))$$

From Corollary 3.2.6 we therefore immediately obtain

Corollary 3.2.10. *For any $2 \leq p \leq n$ one has a natural isomorphism*

$$H^1(U_X, T_{U_X})_{m+(p-1)d} \cong H^{p-1}(U_X, \bigwedge^{p-1} T_{U_X})_{m+d}$$

Lemma 3.2.11. *Let $0 \leq p \leq n$. If $n \neq 2p$ then we have a natural isomorphism*

$$H^p(U_X, \bigwedge^p T_{U_X})_m \cong H^p(X, \bigwedge^p T_X(m))$$

Proof. We know from 3.2.3 that $H^p(U_X, \bigwedge^p T_{U_X})_m \cong H^{n-p,p}_{\text{prim}}(X)$. But for a smooth hypersurface in $\mathbb{P}^{n+1}$ one has $H^{p,n-p}(X) = H^{p,n-p}_{\text{prim}}(X)$ for any p such $n \neq 2p$, due to Hard Lefschetz combined with the Lefschetz hyperplane theorem. □

Lemma 3.2.12. *Let $0 \leq p \leq n$. If $n \neq 2p$ then we have a natural isomorphism*

$$H^{p-1}(\bigwedge^{p-1} T_X(m+d)) \cong H^p(\bigwedge^p T_X(m))$$

Proof. From the short exact sequence

$$0 \to \bigwedge^{p} T_X \to \bigwedge^{p} T_{\mathbb{P}^{n+1}}|_X \to \bigwedge^{p-1} T_X(d) \to 0,$$

using the duality isomorphisms $\bigwedge^i T_X(m) \cong \Omega_X^{n-i}$ and $\bigwedge^i T_{\mathbb{P}^{n+1}}|_X(m) \cong \Omega_{\mathbb{P}^{n+1}}^{n+1-i}|_X(d)$ we get to the long exact sequence

$$\ldots \to H^{n-p,p-1}(X) \to H^{p-1}(X, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d)|_X) \to H^{p-1}(X, \Omega_X^{n-p+1}(d)) \to$$

$$\to H^{n-p,p}(X) \to H^p(X, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d)|_X) \to 0,$$

where the last zero comes from $H^p(X, \Omega_X^{n-p+1}(d)) = 0$ by Kodaira Vanishing.
Now, consider the short exact sequence (see [15])

$$0 \to \Omega_{\mathbb{P}^{n+1}}^{n-p+1} \to \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d) \to \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d)|_X \to 0.$$

This induces the long exact sequence

$$\cdots \to H^p(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d)) \to H^p(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d)|_X) \to$$

$$\to H^{p+1}(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}) \to H^{p+1}(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d)) \to \cdots$$

By Kodaira vanishing we have $H^{p+1}(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d)) = 0$. Also we have the long exact sequence

$$\cdots \to H^{p-1}(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d)) \to H^{p-1}(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d)|_X) \to$$

$$\to H^p(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}) \to H^p(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d)) \to \cdots$$

By Bott vanishing also

$$H^p(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d)) = 0$$

and

$$H^{p-1}(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d)) = 0.$$

Now we consider two subcases. If $n \neq 2p-1$, then

$$H^p(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}(d)|_X) \cong H^{p+1}(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}^{n-p+1}) = 0$$

and

$$H^{p-1}(\mathbb{P}^{n+1}, \Omega^{n-p+1}_{\mathbb{P}^{n+1}}(d)|_X) \cong H^p(\mathbb{P}^{n+1}, \Omega^{n-p+1}_{\mathbb{P}^{n+1}}) = 0$$

(where we used that by hypothesis $n \neq 2p$). So, we find

$$H^{p-1}(X, \Omega_X^{n-p+1}(d)) \cong H^p(X, \Omega_X^{n-p})$$

i.e.,

$$H^{p-1}(X, \bigwedge^{p-1} T_X(m+d)) \cong H^p(X, \bigwedge^{p} T_X(m))$$

in this case. If $n = 2p-1$ then we still have

$$H^p(X, \Omega^p_{\mathbb{P}^{2p}}(d)|_X) = H^p(\mathbb{P}^{2p}, \Omega^p_{\mathbb{P}^{2p}}(d)|_X) \cong H^{p+1}(\mathbb{P}^{2p}, \Omega^p_{\mathbb{P}^{2p}}) = 0$$

while

$$H^{p-1}(X, \Omega^p_{\mathbb{P}^{2p}}(d)|_X) = H^{p-1}(\mathbb{P}^{2p}, \Omega^p_{\mathbb{P}^{2p}}(d)|_X) \cong H^p(\mathbb{P}^{2p}, \Omega^p_{\mathbb{P}^{2p}}) \cong \mathbb{C}.$$

and so our initial long exact sequence becomes

$$\ldots \to H^{p-1,p-1}(X) \xrightarrow{\eta} H^{p,p}(\mathbb{P}^{2p}) \to H^{p-1}(X, \Omega^p_X(d)) \to H^{p-1,p}(X) \to 0.$$

By precomposing the map η with the restriction morphism $H^{p-1,p-1}(\mathbb{P}^{2p}) \to H^{p-1,p-1}(X)$ one obtains the cup product with $c_1(\mathcal{O}_{\mathbb{P}^{2p}}(d))$, which is an isomorphism from $H^{p-1,p-1}(\mathbb{P}^{2p}) \to H^{p,p}(\mathbb{P}^{2p})$. Hence η is surjective, and so

$$H^{p-1}(X, \Omega^p_X(d)) \cong H^{p-1,p}(X).$$

Therefore,

$$H^{p-1}(X, \bigwedge^{p-1} T_X(m+d)) \cong H^p(X, \bigwedge^{p} T_X(m)).$$

in this case, too. $\square$

Corollary 3.2.13. *For any $2 \leq p \leq n$ with $n \neq 2p$ one has a natural isomorphism*

$$H^1(U_X, T_{U_X})_{m+(p-1)d} \cong H^p(U_X, \bigwedge^p T_{U_X})_m$$

We are now left with considering the $n = 2p$ case.

Lemma 3.2.14. *Assume $n = 2p$. Then we have a natural short exact sequence*

$$0 \to H^{p-1}(X, \bigwedge^{p-1} T_X(m+d)) \to H^{p,p}(X) \to \mathbb{C} \to 0,$$

Proof. Reasoning as in the proof of 3.2.11 we get the long exact sequence

$$\ldots \to H^{p-1}(X, \Omega^{p+1}_{\mathbb{P}^{2p+1}}(d)|_X) \to H^{p-1}(X, \Omega^{p+1}_X(d)) \to H^{p,p}(X) \to H^p(X, \Omega^{p+1}_{\mathbb{P}^{n+1}}(d)|_X) \to 0,$$

and we have

$$H^{p-1}(\mathbb{P}^{2p+1}, \Omega^{p+1}_{\mathbb{P}^{2p+1}}(d)|_X) \cong H^p(\mathbb{P}^{2p+1}, \Omega^{p+1}_{\mathbb{P}^{2p+1}}) = 0$$

and

$$H^p(\mathbb{P}^{2p+1}, \Omega^{p+1}_{\mathbb{P}^{2p+1}}(d)|_X) \cong H^{p+1}(\mathbb{P}^{2p+1}, \Omega^{p+1}_{\mathbb{P}^{2p+1}}) \cong \mathbb{C}.$$

□

Corollary 3.2.15. *Assume $n = 2p$, with $p \geq 2$. Then there exists an isomorphsim*

$$H^1(U_X, T_{U_X})_{m+(p-1)d} \cong H^p(U_X, \bigwedge^p T_{U_X})_m$$

Proof. By 3.2.9 we have an isomorphism $H^1(U_X, T_{U_X})_{m+(p-1)d} \cong H^{p-1}(U_X, \bigwedge^{p-1} T_{U_X})_{m+d}$, so we need only to exhibit an isomorphism $H^{p-1}(U_X, \bigwedge^{p-1} T_{U_X})_{m+d} \cong H^p(U_X, \bigwedge^p T_{U_X})_m$. To do this, recall the isomorphism $H^p(U_X, \bigwedge^p T_{U_X})_m \cong H^{p,p}_{\text{prim}}(X)$ from 3.2.3, the short exact sequence

$$0 \to H^{p,p}_{\text{prim}}(X) \to H^{p,p}(X) \xrightarrow{\lambda} H^{p+1,p+1}(X) \to 0$$

coming from Hard Lefschetz, and the fact that $H^{p+1,p+1}(X) \cong \mathbb{C}$ from the Lefschetz hyperplane theorem combined with Hard Lefschetz. Therefore we have a natural short exact sequence

$$0 \to H^p(U_X, \bigwedge^p T_{U_X})_m \to H^{p,p}(X) \to \mathbb{C} \to 0,$$

and we use Lemma 3.2.14 and Corollary 3.2.6 to conclude. □

Putting all the above results toghether we obtain the following

Theorem 3.2.16. *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth degree d projective hypersurface, with $\dim X = n \geq 3$. Then we have*

$$H^1(U_X, T_{U_X})_{pd-n-2} \cong H^p(U_X, \bigwedge^p T_{U_X})_{d-n-2}$$

for every $0 \leq p \leq \dim X$.

Proof. Since $d-n-2$ is precisely the integer m such that $\omega_X \cong \mathcal{O}_X(m)$ by adjunction, for $p \geq 2$ the result follows from Corollary 3.2.15 and Corollary 3.2.13. For $p=1$ there is nothing to prove. Finally, for $p=0$ we have to show that

$$H^1(U_X, T_{U_X})_{-n-2} \cong H^0(U_X, \mathcal{O}_{U_X})_{d-n-2}$$

On the right hand side we have $H^0(X, \mathcal{O}_X(d-n-2))$, while on the left hand side we consider the short exact sequence

$$0 \to \operatorname{coker}\{\lambda_{n-1,0}(-d)\} \to H^1(U_X, T_{U_X})_{-n-2} \to \ker\{\lambda_{n-1,1}(-d)\} \to 0$$

where $\lambda_{i,j}(-d)$ is the Lefschetz morphism $H^j(X, \Omega_X^i(-d)) \to H^{j+1}(X, \Omega_X^{i+1}(-d))$. Now we have $H^1(X, \Omega_X^n(-d)) = H^1(X, \mathcal{O}(-n-2)) = 0$ and $H^2(X, \Omega_X^n(-d)) = H^2(X, \mathcal{O}_X(-n-2)) = 0$, since X is arithmetically Cohen-Macaulay. So the above short exact sequence gives

$$H^1(U_X, T_{U_X})_{-n-2} \cong H^1(X, T_X(-n-2))$$

and to conclude the proof of the theorem we only need to show that

$$H^1(X, T_X(-n-2)) \cong H^0(X, \mathcal{O}_X(d-n-2)).$$

This follows from the normal sheaf exact sequence as in Lemma 3.2.7. Namely from the short exact sequence

$$0 \to T_X \to T_{\mathbb{P}^{n+1}}|_X \to \mathcal{O}_X(d) \to 0,$$

Using the duality isomorphisms $T_X(d-n-2) \cong \Omega_X^{n-1}$ and $T_{\mathbb{P}^{n+1}}|_X(d-n-2) \cong \Omega^n_{\mathbb{P}^{n+1}}|_X(d)$ we get to the sequence

$$\ldots \to H^0(X, \Omega_X^{n-1}(-d)) \to H^0(X, \Omega^n_{\mathbb{P}^{n+1}})|_X) \to H^0(X, \Omega_X^n)) \to$$

$$\to H^1(X, \Omega_X^{n-1}(-d)) \to H^1(X, \Omega^n_{\mathbb{P}^{n+1}})|_X) \to 0,$$

where the last zero comes from $H^1(X, \Omega_X^n) = H^1(X, \mathcal{O}_X(d-n-2)) = 0$ since X is arithmetically Cohen-Macaulay.
Now, consider the short exact sequence (see [15])

$$0 \to \Omega^n_{\mathbb{P}^{n+1}}(-d) \to \Omega^n_{\mathbb{P}^{n+1}} \to \Omega^n_{\mathbb{P}^{n+1}}|_X \to 0.$$

This induces the long exact sequence

$$\cdots \to H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}) \to H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}|_X) \to$$
$$\to H^2(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(-d)) \to H^2(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}) \to \cdots$$

Since $n \geq 3$, we have $H^2(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}) = 0$. Also we have the long exact sequence

$$\cdots \to H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}) \to H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}|_X) \to$$
$$\to H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(-d)) \to H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}) \to \cdots$$

Since $n \geq 3$, we also have
$$H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}) = 0$$

and
$$H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}) = 0.$$

Therefore, we get

$$H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}|_X) \cong H^2(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(-d)) = 0$$

and
$$H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}|_X) \cong H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(-d)) = 0$$

where the rightmost zeroes come from by Bott vanishing. So, we find

$$H^0(X, \Omega^n_X) \cong H^1(X, \Omega^{n-1}_X(-d))$$

i.e.,
$$H^0(X, \mathcal{O}_X(d-n-2) \cong H^1(X, T_X(-n-2))$$

□

Now, why is this interesting? Thanks to the computation above, we have

$$H^{p,n-p}_{\text{prim}}(X) \cong H^p(U_X, \bigwedge^p T_{U_X})_m \cong H^1(U_X, T_{U_X})_{pd-n-2}.$$

On the other hand we know that for a degree d smooth projective hypersurface defined by the polynomial f we have

$$H^1(U_X, T_{U_X}) \cong T^1_{A_X} \cong R_f[d].$$

Therefore, we recover the following result from Griffiths' residue theory [64].

Corollary 3.2.17. *Let $X \subseteq \mathbb{P}^{n+1}$ be a degree d smooth projective hypersurface with $\dim X \geq 3$, defined by the polynomial f. Then we have*

$$R_f[d] \cong H^{p,n-p}_{\text{prim}}(X)$$

for every $0 \leq p \leq \dim X$.

The ideas of this last subsection will be one of the keys for the final chapters of this thesis.

Chapter 4

An application: Hodge theory and deformation of $\mathbb{Q}$-Fano threefolds

We want now to apply the results of the previous chapter to the case of Fano 3-folds with $\mathbb{Q}$-factorial terminal singularities. In the first chapter of this thesis we already investigated the codimension one case. We want now improve our analysis using the tools we have just developed.
According to [76] (following [73] in the case of Mori–Fano 3-folds), the classification of Fano 3-folds consists of finitely many deformation familes. The Hilbert series of members of those families whose generic element lies in codimension at most 4 are known [2, 4] and available on the Graded Ring Database [25]. They fall into $95+85+70+145 = 395$ cases, according to codimension. There may be more than one irreducible family for any given Hilbert series, and in codimension 4 there are usually two or more families in each case [26]; the different families are distinguished by the Euler characteristic of their general member.

4.0.1 The Hodge numbers of Fano 3-folds

The Hodge diamond of a $\mathbb{Q}$-Fano 3-fold X has the form

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 0 & & h^{2,1} & & h^{1,2} & & 0 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$$

The Euler characteristic $e(X)$ of X can be expressed as

$$e(X) = 2 + 2h^{1,1}(X) - 2h^{2,1}(X).$$

In this chapter we calculate these three integers for Fano 3-folds X lying in the known deformation families of Fano 3-folds that have small anticanonical embedding dimension. We explain the different strategies we employ in §4.0.3 below.

The answer is well known in codimension 1, thanks to Griffiths theory. In codimensions 2, 3 and 4, the Euler characteristic is known in many cases by [26], so knowing $h^{1,1}(X)$ completes the calculation. We calculate codimension 2 using our methods described below, and Theorem 4.1.2 is crucial in the higher codimension, non-complete intersection cases—and the cases with higher Picard rank in §4.2.3 use these in an essential way. Thus the first observation is that this is readily computed in low codimension, since every Fano 3-fold in codimension up to 3 appears in one of the two situations of the theorem.

Theorem 4.0.1. *If X is a quasismooth Fano 3-fold that is either*

1. *a complete intersection in weighted projective space, or*
2. *a complete intersection in a weighted cone over a weighted* $\mathrm{Gr}(2,5)$*,*

then $h^{1,1}(X) = 1$.

Proof. We prove that $T^2_{A_X}(-1) = 0$, where A_X is the affine cone on X. This is enough since $H^2(X, K_X) = 0$ allows us to apply 3.1.7, which says $H^{1,1}_{\mathrm{prim}}(X) = T^2_{A_X}(-1) = 0$, and so $h^{1,1}(X) = 1$.

In part (1), the vanishing is [108, 1.3]. For part (2), $T^2_{A_X}(-1) \cong H^1(X, N_{X/\operatorname{C}\mathbb{P}}(-1))$, where $\operatorname{C}\mathbb{P}$ denotes the ambient projective space for the Grassmannian in its Plücker embedding with the addition of the cone variables. From [109, §D.1, Lemma D.3] the flag of schemes $X \subset \operatorname{C}\operatorname{Gr} \subset \operatorname{C}\mathbb{P}$ determines a sequence of sheaves on X:

$$0 \to N_{X/\operatorname{C}\operatorname{Gr}} \to N_{X/\operatorname{C}\mathbb{P}} \to N_{\operatorname{C}\operatorname{Gr}/\operatorname{C}\mathbb{P}} \to 0,$$

where the last map is exact since $H^1(N_{X/\operatorname{C}\operatorname{Gr}}) = 0$. Twisting by $\mathcal{O}_X(-1)$ we get

$$H^1(N_{X/\operatorname{C}\mathbb{P}}(-1)) \cong H^1(N_{\operatorname{C}\operatorname{Gr}/\operatorname{C}\mathbb{P}}(-1)) = 0.$$

This proves part (2). □

Part (1) of this result appeared in a recent preprint, [92], and we found (2) stated several times in the literature, such as [75], but we could not find a proof to cite. In this situation, one would like simply to apply a weighted Lefschetz hyperplane theorem, but unfortunately the linear systems we cut by to make X are rarely base-point free when there are nontrivial weights, so results such as [95, Theorem 1] and [66, Corollary 2.8] do not apply directly.

4.0.2 Fano 3-folds and projection

Consider the following arrangement of projective 3-folds:

$$\begin{array}{ccc} & \widetilde{Y} & \to & X \\ & \downarrow & & \\ Y & \rightsquigarrow & \overline{Y} & \end{array} \tag{4.1}$$

where X and Y are quasismooth, $Y \rightsquigarrow \overline{Y}$ is a degeneration to a singular orbifold whose only non-quasismooth points are ordinary nodes, $\overline{Y} \leftarrow \widetilde{Y}$ is a projective small resolution of the nodes, and $\widetilde{Y} \to X$ is the contraction of a divisor $\widetilde{D} \subset \widetilde{Y}$. The passage from Y to $\widetilde{Y}$, that shrinks a number of vanishing cycles to nodes and then resolves the nodes by exceptional $\mathbb{P}^1$s, is known as a *conifold transition*.

In our context, the exceptional divisor $\widetilde{D} \cong \mathbb{P}(a, b, c)$ maps to a divisor $\mathbb{P}(a, b, c) \to D \subset \overline{Y}$, and the nodes of $\overline{Y}$ lie on D. The small resolution is the relatively $\widetilde{D}$-ample resolution, so is projective, and $\widetilde{D} \to D$ is birational—often an isomorphism, in fact. With this setup, we recall from Clemens [39] (see also [98, §5]):

Theorem 4.0.2 ([39, 98])**.** *Let X and Y be Fano 3-folds related as in diagram (4.1). Then*

$$e(X) = e(Y) + 2n - 2, \tag{4.2}$$

where n is the number of nodes of $\overline{Y}$. In particular, if $h^{1,1}(X) = h^{1,1}(Y)$, then

$$h^{2,1}(X) = h^{2,1}(Y) - n + 1. \tag{4.3}$$

The relevance of this is as follows (see [40, 2.6.3], [26, 3.2]). If X is a Fano 3-fold in codimension k, then it often happens that the *Gorenstein projection* from a quotient singularity sits in diagram (4.1) as $X \dashrightarrow \overline{Y}$, and that $\overline{Y}$ lies in codimension $< k$. If this nodal Fano $\overline{Y}$ deforms to a quasismooth Y whose Hodge numbers are known, then we may recover the invariants of X.

4.0.3 An overview of the calculations

We adopt different tactics to compute the Hodge numbers of a Fano 3-fold X according to its graded ring. When X is a hypersurface, this calculation is well known (see §4.0.1). If X is a complete intersection in weighted projective space or inside a weighted Grassmannian, then $h^{1,1}(X) = 1$ (Theorem 4.0.1). If X arises by (possibly multiple) unprojection from a hypersurface, then we can compute $e(X)$ and hence the whole Hodge diamond. This applies to most X that lie in codimension 2 or 3; see §§4.2.1–4.2.2. Up to codimension 3, this calculation can be done by hand—the key point is to confirm the existence of a nodal degeneration.

Denoting the affine cone over X by A_X, 3.1.7 gives

$$H^{2,1}(X) \cong \left(T^1_{A_X}\right)_{-1}.$$

If X is given by explicit equations, we may use standard algorithms and implementations in computer algebra to calculate $h^{2,1}(X)$, as in the previous chapter. The computer system Singular [51] can do these calculations, but we use Ilten's package [70] for the computer system Macaulay2 [62] since it handles the gradings automatically; with Singular one must pick out the graded piece given generators for the whole module.

In these cases we compute $h^{2,1}(X)$ for a single quasismooth member of each family, expressed in the format we expect. Since $h^{p,q}$ are deformation invariants for orbifolds (since Steenbrink [111, Theorem 2] applies in the context of V-manifolds), the numbers we obtain are also the Hodge numbers of any orbifold Fano 3-fold in the family. (In practice, this computation works when the equations are fairly sparse, and most deformation families in low codimension have quasismooth representatives whose equations have few monomials.)

By 3.1.7,

$$H^{1,1}_{\mathrm{prim}}(X) \cong \left(T^2_{A_X}\right)_{-1},$$

and so if X is given by explicit equations we may compute $h^{1,1}(X)$; see Section 4.2.3 for an example. This algorithm seems to be more complicated, and in practice chosing good equations is delicate.

4.1 Moduli of Fano 3-folds

We explain a relation between $H^{2,1}(X)$ of a Fano threefold X and the tangent space to its versal deformation space $H^1(X, T_X)$. Since deformations of quasismooth Fano 3-folds X are unobstructed (by [106, Theorem 1.7]), this is the number of moduli of X. In fact recall that for a general orbifold the tangent space to the moduli space is $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$. However, for a Gorenstein orbifold with at worst terminal singularities $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \cong H^1(T_X)$ (see Proposition A.4.2 in [45]). Moreover, following our standard notation, $T_X \cong j_* T_{X_0}$.

4.1.1 Deforming a Fano with an elephant

The key motivation comes from Calabi–Yau 3-folds V. In that case, it follows by standard Serre duality (non-canonically, involving a choice of determinant) that $H^{2,1}(V) \cong H^1(V, T_V)$; or one may observe that both are isomorphic to the same graded piece $T^1(A_V)_0 \subset T^1(A_V)$.

If an index 1 Fano 3-fold X has a K3 elephant $E = (x = 0) \subset X$, we may regard the pair (X, E) as a log Calabi–Yau and hope to mimic this relationship. In this case, one has $H^{2,1}(X) \cong T^1(A_X)_{-1}$ and $H^1(X, T_X) \cong T^1(A_X)_0$, and the analogue to the Calabi–Yau isomorphism is the multiplication map $x\colon H^{2,1}(X) \to H^1(X, T_X)$. Of course this map need not be an isomorphism, and in general is not, but Theorem 4.1.2 below explains the difference in terms of the geometry of E. To make this intuition precise, we start with a more general lemma about Fano 3-folds of arbitrary index $m > 0$.

Lemma 4.1.1. *Let X a Fano threefold of index m with $-K_X \overset{\text{lin}}{\sim} mH$, for a $\mathbb{Q}$-Cartier divisor H, and consider (X, H) as a subcanonical pair.*

If $E \subset X$ a K3 elephant $E \in |-K_X|$, then

$$h^1(X, T_X) - h^0(X, T_X) = \alpha_E + h^{2,1}(X) - h^{2,2}(X),$$

where $\alpha_E = h^{1,1}(E) - g_X - 1 = h^{1,1}(E) - h^0(E, \mathcal{O}_E(m))$.

Proof. Consider the standard exact sequence of $\mathcal{O}_X$-modules twisted by $\Omega^2(m)$,

$$0 \to \Omega^2_X \to \Omega^2_X(m) \to \Omega^2_X(m)|_E \to 0.$$

In cohomology this yields a long exact sequence

$$\begin{aligned} 0 \to H^0(\Omega^2_X(m)) \to H^0(\Omega^2(m)_X|_E) \to H^1(\Omega^2_X) \\ \to H^1(\Omega^2_X(m)) \to H^1(\Omega^2_X(m)|_E) \to H^2(\Omega^2_X) \to 0, \end{aligned} \tag{4.4}$$

where $H^0(\Omega^2_X) = 0$ and $H^2(\Omega^2_X(m)) = 0$ by Akizuki–Kodaira–Nakano vanishing.

On the other hand the relative exact tangent sequence

$$0 \to T_E \to T_X|_E \to \mathcal{O}_E(m) \to 0$$

yields a long exact sequence

$$0 \to H^0(E, T_X|_E) \to H^0(E, \mathcal{O}_E(m)) \to H^1(E, T_E) \to H^1(E, T_X|_E) \to 0, \tag{4.5}$$

where $H^1(E, \mathcal{O}_E(m)) = 0$ and $H^0(E, T_E) = H^0(E, \Omega^1_E) = 0$, since E is K3 surface. By (4.4) and (4.5) we get

$$\begin{aligned} h^0(X, \Omega^2_X|_E(m)) + h^1(X, \Omega^2_X(m)) + h^{2,2}(X) = \\ h^{2,1}(X) + h^1(X, \Omega^2_X(m)|_E) + h^0((X, \Omega^2_X(m)) \end{aligned} \tag{4.6}$$

and

$$h^1(T_X|_E) - h^0(T_X|_E) = h^1(T_E) - h^0(\mathcal{O}_E(m)).$$

We have $\Omega^2_X(m) \cong T_X$ from the pairing

$$\Omega^1_X \otimes \Omega^2_X \to \omega_X \cong \mathcal{O}_X(-m).$$

So with α_E defined as in the statement, we get

$$h^1(X, T_X) - h^0(X, T_X) = \alpha_E + h^{2,1}(X) - h^{2,2}(X)$$

as required. $\square$

Theorem 4.1.2. *Let X be a Fano 3-fold with K3 elephant $E \subset X$ and α_E as defined in Lemma 4.1.1. If $h^0(X, T_X) = 0$, then*

$$h^1(X, T_X) - h^{2,1}(X) = \alpha_E - h^{2,2}(X).$$

This gives an estimate of the difference between the moduli and Hodge theory of X: when $b_2 = h^{2,2}(X)$ is small, we have a more moduli than $h^{2,1}$, while if $b_2 \gg 0$ the opposite holds.

4.1.2 Automorphisms of Fano 3-folds in Grassmannians

Lemma 4.1.3. *Let X be a Fano 3-fold of index 1. If X is a weighted complete intersection (in its total anticanonical embedding), then $H^0(X, T_X) = 0$.*

Proof. Recall from Flenner [61, Satz 8.11] that if X is an n-dimensional weighted complete intersection, then $H^p(X, \Omega^q_X(t)) = 0$ whenever $p + q < \dim X$ and $t < q - p$.

The lemma follows by setting $q = 2$, $p = 0$, $t = 1$ together with Serre duality $T_X \cong \Omega^2_X(1)$. $\square$

We prove an analogous result for complete intersection in weighted Grassmannians. Our main interest is in Fano 3-folds of index 1 in codimension 3, $X \subset \mathbb{P}(a_0, \dots, a_6)$, most of which arise in this way. We show in Theorem 4.1.6 below that $H^0(X, T_X) = 0$ in this case. We first show the vanishing result in standard (non-weighted) Grassmannians.

Lemma 4.1.4. *Let X a Fano 3-fold of index 1 that is a complete intersection in a cone $V = C\,\mathrm{Gr}(2, n)$, on vertex a linear projective space that is disjoint from X, over a Grassmannian $\mathrm{Gr}(2, n)$ for some $n \geq 5$. Then $H^0(X, T_X) = 0$.*

Proof. We show that $H^0(X, \Omega^2_X(1)) = 0$, which suffices since $T_X \cong \Omega^2_X(1)$ for X a Fano 3-fold of index 1.

We consider the case $V = \mathrm{Gr}(2, n)$ first, with no cone structure. Suppose that $X = (f_1 = \dots = f_c = 0) \subset G = \mathrm{Gr}(2, n)$, and denote $d_i = \deg f_i$. The Koszul complex of $\mathcal{O}_X$-modules for $\mathcal{O}_X$ twisted by $\Omega^2(1)|_G$ is

$$0 \to \Omega^2_G(1-d_1-\cdots-d_c) \to \cdots \to \bigoplus_{i,j,k} \Omega^2_G(1-d_i-d_i-d_k) \to$$
$$\bigoplus_{i,j} \Omega^2_G(1-d_i-d_j) \to \bigoplus_{i} \Omega^2_G(1-d_i) \to \Omega^2_G(1) \to \Omega^2_G(1)|_X \to 0.$$

By [91, Lemma 0.1], $H^p(G, \Omega^2_G(t)) = 0$ for each of $p = 1, 2, 3$ and any $t \le 1$, and also $H^0(G, \Omega^2_G(1)) = 0$. It follows, by splitting the Koszul sequence above into short exact sequences, that

$$H^0(X, \Omega^2_G(1)|_X) = H^1(X, \Omega^2_G(1)|_X) = H^1(X, \Omega^2_G(1-d_i)|_X) = 0. \tag{4.7}$$

The conormal exact sequence of $X \subset G$ is

$$0 \to \bigoplus_{1\le i\le c} \mathcal{O}_X(-d_i) \to \Omega^1_G|_X \to \Omega^1_X \to 0.$$

Taking its second exterior power and twisting by $\mathcal{O}_X(1)$ we get

$$0 \to \bigoplus_{1\le i,j\le c} \mathcal{O}_X(1-d_id_j) \to \bigoplus_{1\le i\le c} \Omega^2_G(1-d_i)|_X \to \Omega^2_G(1)|_X \to \Omega^2_X(1) \to 0.$$

After splitting this into short exact sequences, the vanishing statements in (4.7) show at once that $H^0(X, \Omega^2_X(1)) = 0$, as required.

The proof for a cone is the same, replacing Ω^2_{Gr} by the extension of the pullback of Ω^2_{Gr} to the complement of the vertex, in which X is a complete intersection; this restricts to X as above, and the proof follows. $\square$

The proof of Lemma 4.1.4 suggests that we need a Bott vanishing type of result to extend the vanishing statements to complete intersections in $w\,\mathrm{Gr}(2,5)$. The following lemma gives the precise statement we need.

Lemma 4.1.5. *Let $wG = wGr(2,5)$. Then $H^p(w\,\mathrm{Gr}, \Omega^2_{w\,\mathrm{Gr}}(-k)) = 0$ for $p = 1, 2, 3$ and any $k > 0$.*

Proof. If $A^\bullet_G$ denotes the punctured affine cone over the (weighted or not) Grassmannian, we have the following diagram

$$
\begin{array}{ccccc}
 & & A_G^\bullet & & \\
 & \pi_1 \swarrow & & \pi_2 \searrow & \\
Gr(2,5) & & & & wGr(2,5)
\end{array}
$$

where π_1 and π_2 denotes (respectively) the standard and the weighted $\mathbb{C}^*$ action. We will use repeatedly the vanishing results from [91, Lemma 0.1] for the standard $\mathrm{Gr}(2,5)$.

The grading on the cohomology groups of $A^\bullet$ will be interpreted in terms of local cohomology at the maximal ideal $\mathfrak{m}$ of the vertex of the affine cone A.

Consider the short exact sequence

$$0 \to \pi_1^*\Omega_G^1 \to \Omega_{A^\bullet}^1 \to \mathcal{O}_{A^\bullet} \to 0. \tag{4.8}$$

Since $H^i(G, \mathcal{O}_G(-k)) = 0$ for any $i < \dim(G)$, we have

$$H^1(A^\bullet, \Omega_{A^\bullet}^1)(-k) = H^1(G, \Omega_G^1(-k)) = 0.$$

One as well gets in the same way $H^0(A^\bullet, \Omega_{A^\bullet}^1)(-k) = 0$.
Raising the short exact sequence (4.8) to the second exterior power we have

$$0 \to \pi_1^*\Omega_G^2 \to \Omega_{A^\bullet}^2 \to \pi_1^*\Omega_G^1 \to 0;$$

by the vanishings above this reduces to

$$H^1(A^\bullet, \Omega_{A^\bullet}^2)(-k) = H^1(G, \Omega_G^2(-k)) = 0.$$

If we now taking similar exact sequence for the second projection π_2 we will have

$$0 \to \pi_2^*\Omega_{wG}^1 \to \Omega_{A^\bullet}^1 \to \mathcal{O}_{A^\bullet} \to 0,$$

$$0 \to \pi_2^*\Omega_{wG}^2 \to \Omega_{A^\bullet}^2 \to \pi_2^*\Omega_{wG}^1 \to 0.$$

Putting all together the vanishings we computed above together with $H^0(\mathcal{O}_{wG}(-k)) = 0$ we get exacly

$$H^1(wG, \Omega_{wG}^2(-k)) = H^1(A^\bullet, \Omega_{A^\bullet}^2)(-k) = 0,$$

and in a similar way we have the same for $i = 2, 3$. $\square$

Theorem 4.1.6. *Let X a Fano 3-fold of index 1 that is a complete intersection in a* *weighted cone $C\,\mathrm{Gr}(2,5)$, on vertex a linearly-embedded weighted projective space that is disjoint from X. Then $H^0(X,T_X)=0$.*

Both the lemma and the theorem can be extended to other weighted Grassmannians $w\,\mathrm{Gr}(2,n)$, for $n\geq 5$, using Bott-type vanishing theorems, but we only need the $\mathrm{Gr}(2,5)$ case here.

4.2 Explicit calculations

It takes a few hundred calculations to complete Tables 4.1–4.3 below. In this section, we give illustrative examples of each type.

4.2.1 Codimension 2

There are 85 deformation families of Fano 3-folds in codimension 2 ([68, 37]), each one a complete intersection with $h^{1,1}(X)=1$. The case $X_{2,3}\subset\mathbb{P}^5$ is classical: $e(X)=c_3(T_X)$ can be calculated directly to give $e(X_{2,3})=-36$ and so $h^{2,1}(X_{2,3})=20$. Of the remaining 84 cases, 66 have a Type I projection (see §4.2.1), and a further 10 cases have a Type II_1 projection (see §4.2.1); 8 cases have no projection of either type (see §4.2.1).

66 cases with Type I projection

Consider one of the families of Fano 3-folds of the form $X=X_{a_3+r,a_4+r}\subset\mathbb{P}(1,a,r-a,a_3,a_4,r)$ with $a<r$. The general member has a quotient singularity $\frac{1}{r}(1,a,r-a)$, and admits a Type I projection, as in diagram (4.1), to a hypersurface:

$$\begin{array}{rcl} X & \subset & \mathbb{P}(1,a,r-a,a_3,a_4,r) \\ \pi_r\downarrow & & \\ D\subset(x_3A=x_4B)=\overline{Y} & \subset & \mathbb{P}(1,a,r-a,a_3,a_4), \end{array}$$

where $D=(x_3=x_4=0)=\mathbb{P}(1,a,r-a)$ and π_r is the projection from the final coordinate point of index r. In each one of these 66 cases, the general $\overline{Y}$ is quasismooth away from $n=\deg(A)\deg(B)/(a(r-a))$ nodes that lie on D (by Bertini's theorem), and it admits a $\mathbb{Q}$-smoothing to a general $Y=Y_{a_3+a_4+r}\subset\mathbb{P}(1,a,r-a,a_3,a_4)$. Thus we calculate $e(X)=e(Y)+2n-2$ by (4.2).

Example 4.2.1. Working from the bottom up in diagram (4.1), let $Y_4\subset\mathbb{P}^4$ be a smooth quartic. We know $e(Y_4)=-56$ and $h^{2,1}(Y_4)=30$. Imposing a linear plane $D=\mathbb{P}^2$ on

Y_4 gives, in coordinates x, y, z, t, u of $\mathbb{P}^4$,

$$\mathbb{P}^2 = D = (x = y = 0) \subset \overline{Y}_4 = (Ax = By) \subset P^4,$$

where A, B are general cubic forms. Such $\overline{Y}$ has 9 nodes at $(A = B = 0) \subset D$. The unprojection of $D \subset Y$ is a quasismooth variety $X_{3,3} \subset \mathbb{P}(1^5, 2)$, which has Fano Hilbert series No. 20522. By (4.2) we have $e(X_{3,3}) = e(Y_4)+18-2 = -40$, and so $h^{2,1}(X_{3,3}) = 30$.

This calculation is recorded in Table 4.2, together with the numerical data described here.

10 cases with Type II_1 projection

Again we work from bottom up in diagram (4.1). Thus, for example, to study X whose Hilbert series P_X is no. 6858 in the GRDB [24], we observe from that database (or by hand from the methods of [4]) that the numerics suggest a Type II_1 projection to $\overline{Y}$ with Hilbert series $P_{\overline{Y}}$ no. 5837, whose general member we know to be of the form $Y_{10} \subset \mathbb{P}(1, 1, 2, 2, 2, 3)$. The task in this case is to impose a divisor D onto a special (nodal) member of this family, where the divisor D may be singular, but its normalisation is $\widetilde{D} \cong \mathbb{P}^2$.

Example 4.2.2. Consider $X = X_{4,6} \subset \mathbb{P}(1, 1, 2, 2, 2, 3)$, which has Fano Hilbert series no. 6858 in [24]. As in Example 4.2.1 we work bottom up, first constructing $D \subset \overline{Y}_{10} \subset \mathbb{P}(1, 1, 2, 2, 5)$ and then unprojecting.

In coordinates x, y, z, t, u on $\mathbb{P}(1, 1, 2, 2, 5)$, the finite morphism

$$\begin{array}{ccc} \mathbb{P}^2 \cong \widetilde{D} & \longrightarrow & D \subset \mathbb{P}(1, 1, 2, 2, 5) \\ (a, b, c) & \mapsto & (a, b, c^2, (a - b)c, abc^3 + c^5) \end{array}$$

has image D defined by the 2×2 minors of

$$M = \begin{pmatrix} t & u & (x - y)z & (xy + z)z^2 \\ x - y & (xy + z)z & t & u \end{pmatrix}.$$

The surface D has two singular points, each of which has a length 2 preimage in $\widetilde{D}$: the point $(1 : 1 : 0 : 0 : 0)$ is the pinched image of $(1 : 1 : 0) \in \widetilde{D}$, and $(1 : 1 : -1 : 0 : 0)$ is the image of two points $(1 : 1 : \pm i)$.

A general $\overline{Y}_{10}$ containing this D has 34 nodes, all of which lie on D. (Two lie at the singularities of D, so the preimage in $\widetilde{D}$ of the singular subscheme of $\overline{Y}$ has length 36 on $\widetilde{D}$.)

The unprojection of $D \subset \overline{Y}$ is given by the maximal Pfaffians of the skew 5×5 matrix

$$\begin{pmatrix} x-y & (xy+z)z & t & u \\ & s_0 & 1 & s_1 + A_3 \\ & & s_1 & B_6 \\ & & & zs_0 + C_4 \end{pmatrix} \quad \text{with entries of degrees} \quad \begin{pmatrix} 1 & 4 & 2 & 5 \\ & 2 & 0 & 3 \\ & & 3 & 6 \\ & & & 4 \end{pmatrix}$$

in $\mathbb{P}(1,1,2,2,5,2,3)$ with coordinates x, y, z, t, u, s_0, s_1, where A, B, C may be determined by the unprojection calculus if we wish to know them explicitly. Eliminating u using the linear equation gives $X_{4,6} \subset \mathbb{P}(1,1,2,2,2,3)$, as required. We know $e(Y) = -124$, so conclude that $e(X) = -124 + 2 \cdot 34 - 2 = -58$ and $h^{2,1}(X) = 31$.

This calculation is recorded in Table 4.2, together with the numerical data described here.

8 cases with no projection

Our projection techniques do not work in these cases. We use computer algebra instead.

Example 4.2.3. Consider a quasismooth Fano 3-fold $X_{6,6} \colon (f = g = 0) \subset \mathbb{P}(1,2^3,3^2)$ with Fano Hilbert series number 3508, defined by

$$f = x^6 + y^3 + z^3 + t^3 + u^2 + v^2 \quad \text{and} \quad g = y^2z + z^2t + t^2y + uv.$$

Iten's Macaulay2 package [70] works as follows (compressing blank lines in the output):

```
Macaulay2, version 1.5
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : loadPackage "VersalDeformations"
o1 = VersalDeformations
o1 : Package
i2 : R = QQ[x,y,z,t,u,v,Degrees=>{2,3,4,5,6,7}];
i3 : I = ideal ( x^6 + y^3 + z^3 + t^3 + u^2 + v^2;
       y^2*z + z^2*t + t^2*y + u*v );
o3 : Ideal of R
i4 : CT^1(-1,I)
                2          24
```

```
o4 : Matrix R  <--- R
```

The answer is that $h^{2,1}(X) = \dim T^1_{A_X}(-1) = 24$.

Since X has a K3 elephant $E = (x = 0) \subset X$ with basket $9 \times \frac{1}{2}(1,1)$ quotient singularities, and $h^0(X, T_X) = 0$ by Lemma 4.1.3, we know at this stage from the moduli formula Theorem 4.1.2 that $h^1(X, T_X) = 11$. This can also be calculated directly by Macaulay2 as follows:

```
i5 : CT^1(0,I)
                2        11
o5 : Matrix R  <--- R
```

The answer is that $h^1(X, T_X) = \dim T^1_{A_X}(0) = 11$.

A similar calculation works with $X_{12,14}\colon (f = g = 0) \subset \mathbb{P}(2,3,4,5,6,7)$, with Hilbert series number 37, with, for example,

$$f = x^6 + y^4 + z^3 - u^2 + tv \quad \text{and} \quad g = x^7 + z^2u + xu^2 + zt^2 + v^2.$$

In this case there is no elephant $E \subset X$, so the moduli formula does not apply as stated. However, the Macaulay2 results are that $h^{2,1}(X) = 18$ and $h^1(X, T_X) = 23$, and so in fact the formula holds with "$\alpha_E = 6$", which is the correct number calculated on X from its basket indices and $h^0(X, \mathcal{O}(1)) = 0$.

4.2.2 Codimension 3

There are 70 known deformation families of Fano 3-folds in codimension 3. The complete intersection $X = X_{2,2,2} \subset \mathbb{P}^5$ is classical: the chern class calculation and Lefschetz gives $e(X) = -24$, $\rho_X = 1$ and $h^{2,1}(X) = 14$. The remaining 69 cases are all complete intersections in weighted Grassmannians $w\,\mathrm{Gr}(2,5)$, and so $h^{1,1}(X) = 1$ in every case.

64 cases Type I

We say that a Fano 3-fold X has a *Type I staircase* if it admits a sequence of alternate Type I projections and Q-smoothings to a hypersurface. Concretely, if $X \subset w\mathbb{P}^6$ lies in codimension 3, then the staircase is

$$\begin{array}{ccccc} & & & \widetilde{Y} & \to & X \\ & & & \downarrow & & \\ & \widetilde{Y} & \to & Y \rightsquigarrow & \overline{Y} & \\ & \downarrow & & & & \\ Z \rightsquigarrow & \overline{Z} & & & & \end{array} \tag{4.9}$$

where $X \dashrightarrow \overline{Y} \subset w\mathbb{P}^5$ eliminates a single variable, $Y \subset w\mathbb{P}^5$ is a general $\mathbb{Q}$-smoothing of $\overline{Y}$, and $Y \dashrightarrow \overline{Z}$ is a projection to a nodal hypersurface $\overline{Z} \subset w\mathbb{P}^4$ as in §4.2.1. Counting nodes on $\overline{Y}$ and $\overline{Z}$ and using the formula of Theorem (4.0.2) completes the calculation of $e(X)$ and $h^{2,1}(X)$.

Of the 64 Fano 3-folds in codimension 3 with a Type I projection, 57 have a Type I staircase to a hypersurface.

Example 4.2.4. Consider the family with Hilbert series no. 20523 in [24]. A typical member $X \subset \mathbb{P}(1,1,1,1,1,2,3)$, in coordinates $x_{1\ldots5}, y, z$, is given by the five maximal Pfaffians of a skew 5×5 matrix of forms

$$\begin{pmatrix} x_1 & x_2 & A & D \\ & x_3 & B & E \\ & & C & F \\ & & & z \end{pmatrix} \quad \text{where the entries have degrees} \quad \begin{pmatrix} 1 & 1 & 2 & 2 \\ & 1 & 2 & 2 \\ & & 2 & 2 \\ & & & 3 \end{pmatrix}.$$

It has a quotient singularity $\frac{1}{3}(1,1,2)$ at the z-coordinate point $P_z \in X$.

Projection from that point is calculated by eliminating z from these equations. Doing that leaves the two Pfaffians of degree 3, which define

$$\overline{Y}_{3,3}\colon \left\{ \begin{pmatrix} A & B & C \\ D & E & F \end{pmatrix} \begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix} = \underline{0} \right\} \subset \mathbb{P}(1,1,1,1,1,2).$$

For general degree 2 forms $A,\ldots,F$, the image $\overline{Y}$ has 6 nodes (by Hilbert–Burch) and a $\mathbb{Q}$-smoothing $Y_{3,3}$ which was computed in Example 4.2.1 above. Making the projection from $Y_{3,3}$ as in Example 4.2.1 completes the staircase. In any case, using the result of Example 4.2.1 gives $e(X) = e(Y) + 2\cdot6 - 2 = -40 + 12 - 2 = -30$, and so $h^{2,1}(X) = 17$.

This calculation is recorded in Table 4.3 below, together with the numerical data of the projection which drives the calculation.

Of the remaining 7 cases, 4 have a Type I projection to a family that arises by Type II_1 unprojection from a hypersurface, so again have a staircase, but with a more complicated second step. A fifth case has a Type I projection to the classical family $Y_{2,3} \subset \mathbb{P}^5$, so we may still apply Theorem 4.0.2.

But in two remaining cases, the image of the Type I projection lies in a family whose Hodge numbers were computed using the algorithms for $\dim T^1$; these cases are dependent on computational algebra.

2 cases Type II_1

Of the cases without a Type I projection, two have a Type II_1 projection: $X_{7,8,8,9,10} \subset \mathbb{P}(1,2,3,3,4,4,5)$ has a Type II_1 projection from $\frac{1}{4}(1,1,3)$ and $X_{10\ldots14} \subset \mathbb{P}(1,3,4,5,5,6,7)$ has a Type II_1 projection from $\frac{1}{5}(1,2,3)$. We consider the latter in detail, following Papadakis [90, 4.4].

Consider $D \subset \mathbb{P}(1,3,4,5,6)$ defined by the maximal minors of

$$M_D = \begin{pmatrix} t & v & yz & z^2 \\ y & z & t & v \end{pmatrix}.$$

This D is the image of $\mathbb{P}(1,2,3) \to \mathbb{P}(1,3,4,5,6)$ given by $(a,b,c) \mapsto (a,c,b^2,bc,b^3)$; the normalising variable b is recovered as the ratio of the rows of M_D.

The general hypersurface $\overline{Y}_{18}$ containing D has the form

$$\overline{Y}_{18} = (A_{12}m_{12} + B_{11}m_{13} + 2B_{12}m_{23} + B_{22}m_{24} = 0) \subset \mathbb{P}(1,3,4,5,6),$$

where m_{ij} denotes the minor of M_D involving columns i and j.

The unprojection of $D \subset \overline{Y}_{18}$ is a codimension 3 variety $X \subset \mathbb{P}(1,3,4,5,5,6,7)$, in coordinates x,y,z,t,u,v,w, defined by the maximal Pfaffians of the skew 5×5 matrix

$$\begin{pmatrix} y & z & t & v \\ & -u & -B_{22} & w + B_{12} \\ & & -w + B_{12} & -B_{11} \\ & & & -uz - A_{12} \end{pmatrix}.$$

For example, setting

$$A_{12} = yv + y^3 + x^9, \quad B_{11} = yt + x^8, \quad B_{12} = 0 \quad \text{and} \quad B_{22} = v$$

results in a quasismooth X, and $\overline{Y}_{18}$ whose non-quasismooth locus is defined by the equations

$$\begin{gathered} zt-yv, \quad y^2z-t^2, \quad yz^2-tv, \quad x^9y+y^4+y^2v+2v^2, \quad x^9z-2x^8t-yt^2+yzv, \\ z^3-v^2, \quad x^9t+y^3t+2z^2v+ytv, \quad x^8y^2+y^3t+z^2v, \quad 2x^8yz-x^9v+y^3v-yv^2 \end{gathered}$$

and consists of 22 nodes, all of which lie on $D \subset \overline{Y}_{18}$.

The general $Y_{18} \subset \mathbb{P}(1,3,4,5,6)$ has $e(Y_{18}) = -80$, so $e(X) = -38$ and $h^{2,1}(X) = 21$.

No Type I or II$_1$ projection

The three remaining cases are $X_{12\dots16} \subset \mathbb{P}(1,4,5,5,6,7,8)$, $X_{16\dots20} \subset \mathbb{P}(1,5,6,7,8,9,10)$ and $X_{14\dots18} \subset \mathbb{P}(1,5,5,6,7,8,9)$. The first has only a type IV projection, while the other two do not have any Gorenstein projections at all. We compute T^1 in these cases: we work out the first in detail here; the other two are similar.

Example 4.2.5. A particular $X_{12\dots16} \subset \mathbb{P}(1,4,5,5,6,7,8)$, in coordinates x,y,z,t,u,v,w, is given by the submaximal Pfaffians of the skew 5×5 matrix

$$\begin{pmatrix} y & z & u & v \\ & u & v & y^2+w \\ & & -y^2+w & x^9+yz \\ & & & zt+t^2 \end{pmatrix}$$

in the usual antisymmetric notation. One checks that the scheme defined by those equations is quasismooth. An embedded deformation of X, that is, one given by $(T^1_{A_X})_0$, preserves the Pfaffian format, varying the entries of the matrix. The maximum degree in the format is 10, so it is easy to compute both T^1_0 and $h^{2,1}(X)$ using Macaulay2 [62, 70] or SINGULAR [51]; the results are 23 and 20 respectively.

We verify the moduli formula of Theorem 4.1.2. The basket of X is

$$\mathcal{B}_X = \left\{ \frac{1}{2}(1,1,1), \frac{1}{4}(1,1,3), 2 \times \frac{1}{5}(1,1,4), \frac{1}{5}(1,2,3) \right\}.$$

The K3 elephant $E = (x=0) \subset X$ is the unique member of $|-K_X|$. It has $h^0(\mathcal{O}_E(1)) = 0$ and $h^{1,1}(E) = 20 - \sum r_i - 1$, where the r_i are the indices of singularities of $\mathcal{B}_X$. Thus

$$h^1(T_X) - h^{2,1}(X) = \alpha_E - h^{2,2}(X) = (20-1-3-3\cdot 4) - 1 = 3,$$

which agrees with $23 - 20$.

The other two cases work similarly; in each case $h^{2,1}(X) = 20$.

4.2.3 Codimension 4

All the calculations in codimensions 4 in this section depend on computer algebra: we use Magma [19] to compute examples of the codimension 4 equations by unprojection, and Macaulay2 [62, 70] for the Hodge numbers.

When a Hilbert series is realised by a Fano 3-fold in codimension 4, it frequently happens that there is more than one deformation family of such Fano 3-folds. For 116 of Hilbert series listed in [24] in codimension 4, [26] computes the different families, and observes that they are distinguished by the Euler characteristic of a quasismooth member. However it does not compute the Picard rank of these Fano 3-folds, in part because there is no known format in which they lie as complete intersections, and so we have no Lefschetz theorem to apply directly. But the computational methods of this chapter still apply, in conjunction with the unprojection construction of [26]. We compute a few examples here as first calculations.

Example 4.2.6. Fano Hilbert series 24097. By [26] there are 3 families of Fano 3-folds $Y \subset \mathbb{P}(1^6, 2^2)$ with (typically) two $\frac{1}{2}(1,1,1)$ quotient singularities, each with the Hilbert series No.24097 in [24]. They arise by unprojection of

$$\mathbb{P}^2 = D \subset \overline{Y} \subset \mathbb{P}(1^6, 2),$$

where $D \subset \mathbb{P}(1^6, 2)$ is a linearly embedded plane, and $\overline{Y}$ is defined by the vanishing of Pfaffians of a skew 5×5 matrix of forms of weights

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 1 & 2 \\ & & & 2 \end{pmatrix}. \tag{4.10}$$

The three famlies arise as so-called "Tom" and "Jerry" unprojections (see [26, §2.3] for details), and the three different results are listed in the Big Table [27]: Tom_1, Jer_{12} and Jer_{15}. Takagi's analysis [113, Theorem 0.3] of prime Fano 3-folds with index 2 terminal singularities shows that the first and third of these families have $h^{1,1}(X) = 1$. Using the Macaulay2 computation, and Theorem 4.1.2 (which holds since each unprojection does indeed carry a quasismooth elephant E with $\alpha_E = 19 - 1 - 5 = 13$), we complete the table below.

unproj type	# nodes	e_X	$h^{1,1}(X)$	$h^{2,1}(X)$	$h^1(X, T_X)$	$h^0(X, T_X)$
Tom_1	6	-14	1	9	21	0
Jer_{12}	8	-10	3	9	19	0
Jer_{15}	9	-12	1	8	20	0

For example, the Jer_{12} case above uses $\overline{Y}$ defined by Pfaffian matrix

$$\begin{pmatrix} t & u & v & w \\ & v & t+u & ux \\ & & x & y^2 - z^2 \\ & & & yz + t^2 + u^2 \end{pmatrix}$$

in the coordinates x, y, z, t, u, v and w of $\mathbb{P}(1^6, 2)$. Such $\overline{Y}$ contains the plane $D = (t = u = v = w = 0)$. Unprojecting $D \subset \overline{Y}$ gives $X \subset \mathbb{P}(1^6, 2^2)$, defined by

$$\begin{gathered} xt - tu - u^2 + v^2, \qquad y^2t - z^2t - xu^2 + vw, \qquad yzt + t^3 + tu^2 - xuv + tw + uw, \\ yzu + t^2u + u^3 - y^2v + z^2v + xw, \qquad x^2u - y^2u + z^2u - xu^2 + yzv + t^2v + u^2v + vw, \\ x^2v - xw + ts, \qquad -xyz - xt^2 - xu^2 - xw - us, \qquad -x^3 + xy^2 - xz^2 + x^2u + vs, \\ x^2y^2 - y^4 - x^2z^2 + 3y^2z^2 - z^4 + yzt^2 - xy^2u + xz^2u + yzu^2 + \\ \qquad\qquad + y^2uv - z^2uv + xtuv + yzw - xuw - tuw + u^2w - ws \end{gathered}$$

in coordinates x, y, z, t, u, v, w and unprojection variable s.

Example 4.2.7. Fano Hilbert series 24078. By [26] there are 3 families of Fano 3-folds $X \subset \mathbb{P}(1^6, 2, 3)$ with (typically) two $\frac{1}{3}(1, 1, 2)$ quotient singularities, each with the Hilbert series No.24078 in [24]. They arise by unprojection of

$$\mathbb{P}^2 = D \subset \overline{Y} \subset \mathbb{P}(1^6, 2),$$

where $D \subset \mathbb{P}(1^6, 2)$ is a linearly embedded $\mathbb{P}(1, 1, 2)$, and $\overline{Y}$ is defined by the vanishing of Pfaffians of a skew 5×5 matrix of forms of the same weights as (4.10) above.

The three different results [27] are: Tom_1, Tom_5 and Jer_{12}. In this case the elephant $E \subset X$ has $\alpha_E = 13$, and the table below summarises the results.

unproj type	# nodes	e_X	$h^{1,1}(X)$	$h^{2,1}(X)$	$h^1(X, T_X)$	$h^0(X, T_X)$
Tom_1	5	-16	1	10	22	0
Tom_5	4	-18	2	12	23	0
Jer_{12}	6	-14	1	9	21	0

These calculations seem to be on the limit of what we can do, as they terminate only when the equations are relatively small. For example, the Tom_5 case above uses $\overline{Y}$ defined by Pfaffian matrix

$$\begin{pmatrix} z & t & v+u & w \\ & u & t & xv+zu \\ & & z & w-y^2 \\ & & & x^2-v^2 \end{pmatrix}$$

in the coordinates x, y, z, t, u, v and w of $\mathbb{P}(1^6, 2)$.

Of the 145 Hilbert series of Fano 3-folds listed in [24] as presented naturally in codimension 4, 116 have the numerical properties consistent with having a Type I unprojection. The unprojection analysis of these is the subject of [26], with the results presented in [27], and in principle they could all be computed as above. A further 16 Hilbert series have the numerical properties of a Type II_1 projection, and a computational approach following Papadakis [90] is conceivable.

Some of the remaining 13 cases have more complicated projections that we do not know how to work with systematically yet, but four cases have no Gorenstein projections at all, and some other approach is required (even to write down examples by equations). These cases are:

No. 25 $\quad X \subset \mathbb{P}(2,5,6,7,8,9,10,11)$

No. 166 $\quad X \subset \mathbb{P}(2,2,3,3,4,4,5,5)$

No. 282 $\quad X \subset \mathbb{P}(1,6,6,7,8,9,10,11)$

No. 308 $\quad X \subset \mathbb{P}(1,5,6,6,7,8,9,10)$.

4.2.4 A quasismooth unprojection from codimension 4

We construct a codimension 4, quasismooth Fano 3-fold $X \subset \mathbb{P}(1^6, 2^2)$ with Hilbert series number 24097 which contains a quasismooth divisor $E \subset X$ that is itself a complete intersection. We adapt Example 4.2.6 so that the codimension 3 projection $Y \subset \mathbb{P}(1^6, 2)$ contains two divisors: the coordinate planes $D = \mathbb{P}^2$ and $E = \mathbb{P}(1,1,2)$ meeting along the coordinate line $\mathbb{P}^1$.

Indeed define Y by the maximal Pfaffians of

$$\begin{pmatrix} t & u & v & w \\ & v & u & -zv-u^2 \\ & & z-t & yz-x^2 \\ & & & y^2-t^2 \end{pmatrix}$$

in the coordinates x, y, z, t, u, v and w of $\mathbb{P}(1^6, 2)$. Then $D = (t = u = v = w = 0) = \mathbb{P}^2$ lies inside Y in Jer_{12} format while $E = (z = t = u = v = 0) = \mathbb{P}(1, 1, 2)$ lies inside Y in Tom_5 format.

Altogether Y has 8 nodes; these all lie on D (in accordance with Jer_{12} unprojection of D to construct Hilbert series 24097), and 4 of them lie on the intersection $D \cap E$ (in accordance with the Tom_5 unprojection or E to construct Hilbert series 24078).

We may unproject either divisor, and we choose to unproject $D \subset Y$ to give $X \subset \mathbb{P}(1^6, 2^2)$. All the 8 nodes are resolved by this, and X is quasismooth. The Fano 3-fold X has Picard rank $\rho_X = 3$ (as in Example 4.2.6 above).

Furthermore, $E \subset Y$ has birational image in X, which we also denote $E \subset X$ defined by equations

$$E = (z = t = u = v = 0) \cap X \subset \mathbb{P}(1^6, 2^2),$$

in coordinates x, y, z, t, u, v, w, s. Computing the unprojection shows that $E \cong (x^4 - y^4 - w^2 + ws = 0) \subset \mathbb{P}(1^2, 2^2)$ in coordinates x, y, w, s, which is $\mathbb{P}(1, 1, 2)$ blown up in 4 points on the coordinate line $L = \mathbb{P}(1, 1)$ followed by the contraction of the resulting -2-curve $\widetilde{L}$, the birational transform of L. Thus it is a index 2 Fano surface with two $\frac{1}{2}(1, 1)$ quotient singularities, Picard rank 4 and $K_E^2 = 4$. It can be unprojected to an ordinary, isolated cDV singular point (in new local coordinates, the cone on E) on an otherwise smooth complete intersection $Z_{2,2,2} \subset \mathbb{P}^6$.

4.3 Hodge numbers of Fano 3-folds

Tables 4.1–4.3 in 4.3.3 below list the invariants for all known families of Fano 3-folds in codimension at most 3. The majority of the calculations can be carried out by hand. We use computer algebra where not, and also use it as a check on all results.

In codimensions 1 and 2 respectively the Fano 3-folds come from Iano-Fletcher ([68] Tables 5 and 6 respectively; in codimensions 3 and 4 they are from Altınok ([2]). The graded ring database identifier (denoted 'GRDB' in the tables) is that of [24].

4.3.1 Use of computer algebra

The explicit calculations we need are standard, although sometimes rather involved. There are three places computer algebra may assist.

1. Checking that a variety is quasismooth can usually be done with Bertini's theorem. In codimension 3 and 4, this can be carried out as in [23, §3–4], for example, when Type I projections (and staircases) are available. In other cases, we check the Jacobian condition by machine. This, or some equivalent (such as [114, Theorem 5.5] or [17]), can be checked by computer algebra given explicit equations.

2. Checking that a variety has only ordinary nodes as singularities, and counting those nodes, can again usually be done by Bertini's theorem together with a chern class calculation when we have Type I projections; see for example [23, §4] for the nodes and [26, §7] for the count. In other cases, we use computer algebra following [26, §6].

3. Computing the dimensions of graded pieces of spaces $T^1_{A_X}$ seems too hard by hand in most cases, but there are algorithms to do this based on Gröbner basis.

We are indebted to the developers of the computer algebra systems Macaulay2 [62], Magma [19] and Singular [51] that we used for these calculations, and to Ilten [70] for the Versal Deformation package for Macaulay2. (The latter conveniently handles the gradings on variables automatically when computing graded pieces of $T^1_{A_X}$; on other systems we had to pick out the graded piece given generators for the whole module "by hand".)

In practice, most computations here work when the equations of the Fano 3-fold are fairly sparse, and as the codimension increases it becomes harder to find such sparse representatives.

4.3.2 Blache's orbifold formula

Let V be a projective orbifold of dimension n, embedded as a quasismooth subvariety of weighted projective space $V \subset \mathbb{P} = \mathbb{P}(a_0, \dots, a_N)$. We suppose, in addition, that V is a manifold away from a finite set of strictly orbifold points $Q_1, \dots, Q_s \in V$.

We define the orbifold total chern class $c_{\text{orb}}(T_{\mathbb{P}}) = 1 + c_{1,\text{orb}}(T_{\mathbb{P}}) + \cdots + c_{n,\text{orb}}(T_{\mathbb{P}})$ of $\mathbb{P}$ via

$$0 \to \mathcal{O}_{\mathbb{P}} \to \oplus_{i=0}^{N} \mathcal{O}_{\mathbb{P}}(a_i) \to T_{\mathbb{P}} \to 0.$$

Taking the restriction of this to V, we derive the top chern class $c_{\mathrm{orb}}(V)$ of V from the tangent exact sequence

$$0 \to T_V \to T_{\mathbb{P}|V} \to N_{V|\mathbb{P}} \to 0$$

exactly as in the smooth case: that is, we make the formal computation

$$1 + c_{1,\mathrm{orb}}(T_{\mathbb{P}}) + \cdots + c_{\mathrm{orb},N}(T_{\mathbb{P}}) = c_{\mathrm{orb}}(T_{\mathbb{P}}) := \prod(1 + a_i h),$$

where $H^2(\mathbb{P}, \mathbb{Q}) = h\mathbb{Q}$ and $c_{\mathrm{orb},j} \in H^{2j}(\mathbb{P}, \mathbb{Q})$, and then

$$\left(1 + c_{\mathrm{orb},1}(T_V) + \cdots + c_{\mathrm{orb},n}(T_V)\right) c(N_{V|\mathbb{P}}) = c_{\mathrm{orb}}(T_{\mathbb{P}}).$$

Then we define the orbifold euler class $e_{\mathrm{orb}}(V)$ by

$$e_{\mathrm{orb}}(V) := \int_V c_{n,\mathrm{orb}}(V) \in \mathbb{Q}.$$

This is a formal computation that ignores orbifold behaviour. However, it is related to the topological euler characteristic $e(V)$ by the following theorem of Blache [16].

Theorem 4.3.1 ([16] (2.11–14))**.** *Let V be a projective orbifold with finite orbifold locus as above. Then $e_{\mathrm{orb}}(X) \in \mathbb{Q}$ satisfies*

$$e(X) = e_{\mathrm{orb}}(X) + \sum_{Q \in \mathcal{B}} \frac{r-1}{r},$$

where $r = r(Q)$ is the local index of the orbifold point Q.

For a hypersurface $X_d \subset \mathbb{P}(a_0, \ldots, a_{n+1})$ we have

$$e_{\mathrm{orb}}(X) = \text{the coefficient of } h^n \text{ in series expansion of } \left(\frac{\prod(1 + a_i h)}{1 + dh} \deg(X)\right).$$

For example, Fano number 337 is $X_{28} \subset \mathbb{P}(1, 4, 6, 7, 11)$ and has basket

$$\mathcal{B} = \left\{2 \times \frac{1}{2}(1,1,1), \frac{1}{6}(1,1,5), \frac{1}{11}(1,4,7)\right\}.$$

Calculating as above gives

$$
\begin{aligned}
e(X) &= e_{\mathrm{orb}}(X) + 2\times\frac{1}{2}+\frac{5}{6}+\frac{10}{11}\\
&= \mathrm{coeff}_{h^3}\left((1+29h+309h^2)(1-28h+784h^2-21952h^3)\right)\frac{28}{4\cdot 6\cdot 7\cdot 11}\\
&\qquad +2\times\frac{1}{2}+\frac{5}{6}+\frac{10}{11}\\
&= \mathrm{coeff}_{h^3}(1+h+281h^2-6385h^3)\frac{1}{66}+2\times\frac{1}{2}+\frac{5}{6}+\frac{10}{11}\\
&= \frac{-6385}{66}+1+5/6+10/11\\
&= -94.
\end{aligned}
$$

This agrees with our calculation $h^{2,1}(X)=49$ and $e(X)=4-2\times 49$.

4.3.3 Tables of results

Tables 4.1–4.3 list the Hodge number $h^{2,1}(X)$, the topological euler characteristic $e(X)$ and the number of moduli $h^1(T_X)=\dim H^1(X,T_X)$ for quasismooth members X of the families of Fano 3-folds in codimensions 1–3 respectively.

In codimension 1, we apply the Griffith's Residue Theorem in §4.0.1 together with the formulas of Theorem 4.1.2. In codimension 2, Table 4.2 documents the method we use to compute the invariants. This could be the conventional chern class calculation, indicated by $c_3(T_X)$, a computer calculation of $T^1_{A_X}$, indicated by T^1, or a projection calculation, indicated by I or II_1 depending on the type of the projection. Where we use a projection, we also list the centre $\frac{1}{r}$ of projection (leaving the polarising weights of $\frac{1}{r}(1,a,-a)$ implicit), the number of nodes on the image of projection, and the number of that image in the GRDB. Where there is more than one possible centre of projection, we list them all. Combining this data with the results of Table 4.1 and Theorems 4.1.2 and 4.0.2 calculates the invariants. For example, number 25022, $X_{3,3}\subset\mathbb{P}(1^5,2)$ (the second line in Table 4.2) projects to number 20521 with 9 nodes; the Euler charactistic of the smoothed image is listed in Table 4.1 as -56, and so the for $X_{3,3}$ it is $-56+2\times9-2=-40$, as displayed.

In codimension 3, Table 4.3 documents the method we use in the 70 cases as follows:

1. 57 cases have at least one 'staircase' of two Type I projections to a hypersurface. This is indicated by I–I.

2. 4 cases have a Type I projection to a codimension 2 family that has as a Type II_1 projection to a hypersurface (indicated by I–II_1).

3. 2 cases have a Type II_1 projection directly to a hypersurface (II_1).

4. 2 cases have a Type I projection to a codimension 2 family with no projection (I–T^1).

5. 1 case has a Type I projection to a known smooth Fano (I–smooth).

6. 1 case is a known smooth Fano complete intersection ($c_3(T_X)$).

7. 3 cases have no Type I or II_1 projections at all (T^1).

Again, where there is a projection from X we list the centre $\frac{1}{r}$, the number of nodes and the GRDB identifier for each possibility, and applying Theorems 4.1.2 and 4.0.2 together with data from previous tables calculates the invariants.

Table 4.1: Codimension 1: $h^{1,1}(X) = 1$ and $h^0(X, T_X) = 0$ in all cases.

GRDB	variety	$h^{2,1}$	$e(X)$	$h^1(T_X)$
20521	$X_4 \subset \mathbb{P}^4$	30	-56	43
16203	$X_5 \subset \mathbb{P}(1,1,1,1,2)$	38	-72	51
16202	$X_6 \subset \mathbb{P}(1,1,1,1,3)$	52	-100	66
11101	$X_6 \subset \mathbb{P}(1,1,1,2,2)$	41	-78	55
10981	$X_7 \subset \mathbb{P}(1,1,1,2,3)$	51	-98	63
10980	$X_8 \subset \mathbb{P}(1,1,1,2,4)$	64	-124	78
10960	$X_9 \subset \mathbb{P}(1,1,1,3,4)$	71	-138	83
10959	$X_{10} \subset \mathbb{P}(1,1,1,3,5)$	85	-166	98
10958	$X_{12} \subset \mathbb{P}(1,1,1,4,6)$	111	-218	125
5838	$X_8 \subset \mathbb{P}(1,1,2,2,3)$	45	-86	58
5837	$X_{10} \subset \mathbb{P}(1,1,2,2,5)$	64	-124	79
5257	$X_9 \subset \mathbb{P}(1,1,2,3,3)$	49	-94	62
5157	$X_{10} \subset \mathbb{P}(1,1,2,3,4)$	56	-108	66
5153	$X_{11} \subset \mathbb{P}(1,1,2,3,5)$	65	-126	74
5152	$X_{12} \subset \mathbb{P}(1,1,2,3,6)$	75	-146	88
5137	$X_{12} \subset \mathbb{P}(1,1,2,4,5)$	70	-136	81
5136	$X_{14} \subset \mathbb{P}(1,1,2,4,7)$	90	-176	102

Continued on next page

5134	$X_{15} \subset \mathbb{P}(1,1,2,5,7)$	97	-190	106
5133	$X_{16} \subset \mathbb{P}(1,1,2,5,8)$	108	-212	119
5132	$X_{18} \subset \mathbb{P}(1,1,2,6,9)$	128	-252	141
4984	$X_{12} \subset \mathbb{P}(1,1,3,4,4)$	60	-116	73
4909	$X_{13} \subset \mathbb{P}(1,1,3,4,5)$	66	-128	73
4907	$X_{15} \subset \mathbb{P}(1,1,3,4,7)$	82	-160	89
4906	$X_{16} \subset \mathbb{P}(1,1,3,4,8)$	91	-178	102
4893	$X_{15} \subset \mathbb{P}(1,1,3,5,6)$	78	-152	87
4892	$X_{18} \subset \mathbb{P}(1,1,3,5,9)$	104	-204	114
4891	$X_{21} \subset \mathbb{P}(1,1,3,7,10)$	126	-248	133
4890	$X_{22} \subset \mathbb{P}(1,1,3,7,11)$	136	-268	144
4889	$X_{24} \subset \mathbb{P}(1,1,3,8,12)$	154	-304	165
4835	$X_{16} \subset \mathbb{P}(1,1,4,5,6)$	77	-150	83
4834	$X_{20} \subset \mathbb{P}(1,1,4,5,10)$	108	-212	119
4822	$X_{18} \subset \mathbb{P}(1,1,4,6,7)$	88	-172	94
4821	$X_{22} \subset \mathbb{P}(1,1,4,6,11)$	120	-236	127
4820	$X_{28} \subset \mathbb{P}(1,1,4,9,14)$	165	-326	172
4819	$X_{30} \subset \mathbb{P}(1,1,4,10,15)$	182	-360	190
4807	$X_{21} \subset \mathbb{P}(1,1,5,7,8)$	99	-194	104
4806	$X_{26} \subset \mathbb{P}(1,1,5,7,13)$	137	-270	143
4805	$X_{36} \subset \mathbb{P}(1,1,5,12,18)$	211	-418	218
4794	$X_{24} \subset \mathbb{P}(1,1,6,8,9)$	110	-216	115
4793	$X_{30} \subset \mathbb{P}(1,1,6,8,15)$	154	-304	160
4792	$X_{42} \subset \mathbb{P}(1,1,6,14,21)$	240	-476	247
2402	$X_{12} \subset \mathbb{P}(1,2,2,3,5)$	47	-90	59
2401	$X_{14} \subset \mathbb{P}(1,2,2,3,7)$	60	-116	74
1389	$X_{12} \subset \mathbb{P}(1,2,3,3,4)$	40	-76	54
1162	$X_{14} \subset \mathbb{P}(1,2,3,4,5)$	45	-86	52
1160	$X_{16} \subset \mathbb{P}(1,2,3,4,7)$	54	-104	62
1159	$X_{18} \subset \mathbb{P}(1,2,3,4,9)$	65	-126	76
1155	$X_{15} \subset \mathbb{P}(1,2,3,5,5)$	48	-92	60
1149	$X_{17} \subset \mathbb{P}(1,2,3,5,7)$	56	-108	60
1147	$X_{18} \subset \mathbb{P}(1,2,3,5,8)$	61	-118	66
1146	$X_{20} \subset \mathbb{P}(1,2,3,5,10)$	72	-140	82

Continued on next page

Table 4.1 continued from previous page

1144	$X_{21} \subset \mathbb{P}(1,2,3,7,9)$	72	-140	78
1143	$X_{24} \subset \mathbb{P}(1,2,3,7,12)$	89	-174	97
1142	$X_{24} \subset \mathbb{P}(1,2,3,8,11)$	87	-170	93
1141	$X_{26} \subset \mathbb{P}(1,2,3,8,13)$	99	-194	106
1140	$X_{30} \subset \mathbb{P}(1,2,3,10,15)$	121	-238	131
1113	$X_{20} \subset \mathbb{P}(1,2,4,5,9)$	62	-120	70
1112	$X_{22} \subset \mathbb{P}(1,2,4,5,11)$	72	-140	81
1079	$X_{20} \subset \mathbb{P}(1,2,5,6,7)$	55	-106	60
1078	$X_{26} \subset \mathbb{P}(1,2,5,6,13)$	80	-156	87
1076	$X_{27} \subset \mathbb{P}(1,2,5,9,11)$	77	-150	79
1075	$X_{32} \subset \mathbb{P}(1,2,5,9,16)$	100	-196	104
1074	$X_{42} \subset \mathbb{P}(1,2,5,14,21)$	144	-284	150
1067	$X_{30} \subset \mathbb{P}(1,2,6,7,15)$	88	-172	96
866	$X_{15} \subset \mathbb{P}(1,3,3,4,5)$	40	-76	52
545	$X_{18} \subset \mathbb{P}(1,3,4,5,6)$	42	-80	49
539	$X_{19} \subset \mathbb{P}(1,3,4,5,7)$	45	-86	47
537	$X_{20} \subset \mathbb{P}(1,3,4,5,8)$	48	-92	53
536	$X_{24} \subset \mathbb{P}(1,3,4,5,12)$	63	-122	71
534	$X_{24} \subset \mathbb{P}(1,3,4,7,10)$	57	-110	58
533	$X_{28} \subset \mathbb{P}(1,3,4,7,14)$	72	-140	80
532	$X_{30} \subset \mathbb{P}(1,3,4,10,13)$	74	-144	75
531	$X_{34} \subset \mathbb{P}(1,3,4,10,17)$	90	-176	92
530	$X_{36} \subset \mathbb{P}(1,3,4,11,18)$	97	-190	101
529	$X_{42} \subset \mathbb{P}(1,3,4,14,21)$	120	-236	125
508	$X_{21} \subset \mathbb{P}(1,3,5,6,7)$	45	-86	51
507	$X_{33} \subset \mathbb{P}(1,3,5,11,14)$	74	-144	74
506	$X_{38} \subset \mathbb{P}(1,3,5,11,19)$	92	-180	93
505	$X_{48} \subset \mathbb{P}(1,3,5,16,24)$	126	-248	130
500	$X_{24} \subset \mathbb{P}(1,3,6,7,8)$	48	-92	56
356	$X_{24} \subset \mathbb{P}(1,4,5,6,9)$	45	-86	47
355	$X_{30} \subset \mathbb{P}(1,4,5,6,15)$	62	-120	69
353	$X_{25} \subset \mathbb{P}(1,4,5,7,9)$	46	-88	46
352	$X_{32} \subset \mathbb{P}(1,4,5,7,16)$	65	-126	69
351	$X_{44} \subset \mathbb{P}(1,4,5,13,22)$	91	-178	91

Continued on next page

Table 4.1 continued from previous page

350	$X_{54} \subset \mathbb{P}(1,4,5,18,27)$	120	-236	121
337	$X_{28} \subset \mathbb{P}(1,4,6,7,11)$	49	-94	50
336	$X_{34} \subset \mathbb{P}(1,4,6,7,17)$	65	-126	67
296	$X_{27} \subset \mathbb{P}(1,5,6,7,9)$	42	-80	42
295	$X_{30} \subset \mathbb{P}(1,5,6,8,11)$	46	-88	45
294	$X_{38} \subset \mathbb{P}(1,5,6,8,19)$	64	-124	64
293	$X_{66} \subset \mathbb{P}(1,5,6,22,33)$	120	-236	120
289	$X_{40} \subset \mathbb{P}(1,5,7,8,20)$	64	-124	68
271	$X_{36} \subset \mathbb{P}(1,7,8,9,12)$	42	-80	41
270	$X_{50} \subset \mathbb{P}(1,7,8,10,25)$	63	-122	62

Table 4.2: Codimension 2: $h^{1,1}(X) = 1$ and $h^0(X, T_X) = 0$ in all cases.

grdb	variety	method	$\frac{1}{r}$, #nodes, target id	$h^{2,1}$	$e(X)$	$h^1(T_X)$
24076	$X_{2,3} \subset \mathbb{P}^5$	$c_3(T_X)$		20	-36	34
20522	$X_{3,3} \subset \mathbb{P}(1,1,1,1,1,2)$	I	$\frac{1}{2}, 9, 20521$	22	-40	36
16225	$X_{3,4} \subset \mathbb{P}(1,1,1,1,2,2)$	I	$\frac{1}{2}, 12, 16203$	27	-50	41
16204	$X_{4,4} \subset \mathbb{P}(1,1,1,1,2,3)$	I	$\frac{1}{3}, 8, 16203$	31	-58	45
11435	$X_{4,4} \subset \mathbb{P}(1,1,1,2,2,2)$	I	$\frac{1}{2}, 16, 11101$	26	-48	39
11102	$X_{4,5} \subset \mathbb{P}(1,1,1,2,2,3)$	I	$\frac{1}{2}, 20, 10981$; $\frac{1}{3}, 10, 11101$	32	-60	45
11002	$X_{4,6} \subset \mathbb{P}(1,1,1,2,3,3)$	I	$\frac{1}{3}, 12, 10981$	40	-76	53
10983	$X_{5,6} \subset \mathbb{P}(1,1,1,2,3,4)$	I	$\frac{1}{2}, 30, 10960$; $\frac{1}{4}, 10, 10981$	42	-80	55
10982	$X_{6,6} \subset \mathbb{P}(1,1,1,2,3,5)$	I	$\frac{1}{5}, 6, 10981$	46	-88	59
10961	$X_{6,8} \subset \mathbb{P}(1,1,1,3,4,5)$	I	$\frac{1}{5}, 12, 10960$	60	-116	73
6858	$X_{4,6} \subset \mathbb{P}(1,1,2,2,2,3)$	II_1	$\frac{1}{2}, 34, 5837$	31	-58	43
5857	$X_{5,6} \subset \mathbb{P}(1,1,2,2,3,3)$	I	$\frac{1}{3}, 15, 5838$	31	-58	42
5843	$X_{6,6} \subset \mathbb{P}(1,1,2,2,3,4)$	I	$\frac{1}{4}, 12, 5838$	34	-64	45
5839	$X_{6,7} \subset \mathbb{P}(1,1,2,2,3,5)$	I	$\frac{1}{5}, 7, 5838$	39	-74	50
5514	$X_{6,6} \subset \mathbb{P}(1,1,2,3,3,3)$	I	$\frac{1}{3}, 18, 5257$	32	-60	42
5261	$X_{6,7} \subset \mathbb{P}(1,1,2,3,3,4)$	I	$\frac{1}{3}, 21, 5157$; $\frac{1}{4}, 14, 5257$	36	-68	46
5258	$X_{6,8} \subset \mathbb{P}(1,1,2,3,3,5)$	I	$\frac{1}{3}, 24, 5153$; $\frac{1}{5}, 8, 5257$	42	-80	52
5200	$X_{6,8} \subset \mathbb{P}(1,1,2,3,4,4)$	I	$\frac{1}{4}, 16, 5157$	41	-78	51
5161	$X_{7,8} \subset \mathbb{P}(1,1,2,3,4,5)$	I	$\frac{1}{3}, 28, 5137$; $\frac{1}{5}, 14, 5157$	43	-82	53

Continued on next page

5159	$X_{6,9} \subset \mathbb{P}(1,1,2,3,4,5)$	I	$\frac{1}{4}, 18, 5153;\ \frac{1}{5}, 9, 5157$	48	-92	58
5158	$X_{8,9} \subset \mathbb{P}(1,1,2,3,4,7)$	I	$\frac{1}{7}, 6, 5157$	51	-98	61
5156	$X_{6,10} \subset \mathbb{P}(1,1,2,3,5,5)$	I	$\frac{1}{5}, 10, 5153$	56	-108	66
5155	$X_{8,10} \subset \mathbb{P}(1,1,2,3,5,7)$	I	$\frac{1}{3}, 40, 5134;\ \frac{1}{7}, 8, 5153$	58	-112	68
5154	$X_{9,10} \subset \mathbb{P}(1,1,2,3,5,8)$	I	$\frac{1}{8}, 6, 5153$	60	-116	70
5138	$X_{8,10} \subset \mathbb{P}(1,1,2,4,5,6)$	I	$\frac{1}{6}, 16, 5137$	55	-106	65
5135	$X_{10,14} \subset \mathbb{P}(1,1,2,5,7,9)$	I	$\frac{1}{9}, 10, 5134$	88	-172	98
4985	$X_{8,9} \subset \mathbb{P}(1,1,3,4,4,5)$	I	$\frac{1}{4}, 24, 4909;\ \frac{1}{5}, 18, 4984$	43	-82	51
4936	$X_{8,10} \subset \mathbb{P}(1,1,3,4,5,5)$	I	$\frac{1}{5}, 20, 4909$	47	-90	55
4912	$X_{9,10} \subset \mathbb{P}(1,1,3,4,5,6)$	I	$\frac{1}{4}, 30, 4893;\ \frac{1}{6}, 18, 4909$	49	-94	57
4911	$X_{8,12} \subset \mathbb{P}(1,1,3,4,5,7)$	I	$\frac{1}{5}, 24, 4907;\ \frac{1}{7}, 8, 4909$	59	-114	67
4910	$X_{10,12} \subset \mathbb{P}(1,1,3,4,5,9)$	I	$\frac{1}{9}, 6, 4909$	61	-118	69
4908	$X_{12,14} \subset \mathbb{P}(1,1,3,4,7,11)$	I	$\frac{1}{11}, 6, 4907$	77	-150	85
4894	$X_{10,12} \subset \mathbb{P}(1,1,3,5,6,7)$	I	$\frac{1}{7}, 20, 4893$	59	-114	67
4848	$X_{10,12} \subset \mathbb{P}(1,1,4,5,6,6)$	I	$\frac{1}{6}, 24, 4835$	54	-104	61
4837	$X_{11,12} \subset \mathbb{P}(1,1,4,5,6,7)$	I	$\frac{1}{5}, 33, 4822;\ \frac{1}{7}, 22, 4835$	56	-108	63
4836	$X_{12,15} \subset \mathbb{P}(1,1,4,5,6,11)$	I	$\frac{1}{11}, 6, 4835$	72	-140	79
4823	$X_{12,14} \subset \mathbb{P}(1,1,4,6,7,8)$	I	$\frac{1}{8}, 24, 4822$	65	-126	72
4808	$X_{14,16} \subset \mathbb{P}(1,1,5,7,8,9)$	I	$\frac{1}{9}, 28, 4807$	72	-140	78
4795	$X_{16,18} \subset \mathbb{P}(1,1,6,8,9,10)$	I	$\frac{1}{10}, 32, 4794$	79	-154	85
3508	$X_{6,6} \subset \mathbb{P}(1,2,2,2,3,3)$	T^1		24	-44	34
2419	$X_{6,8} \subset \mathbb{P}(1,2,2,3,3,4)$	II_1	$\frac{1}{3}, 33, 2401$	28	-52	37
2409	$X_{6,10} \subset \mathbb{P}(1,2,2,3,4,5)$	II_1	$\frac{1}{4}, 25, 2401$	36	-68	45
2403	$X_{9,10} \subset \mathbb{P}(1,2,2,3,5,7)$	I	$\frac{1}{7}, 9, 2402$	39	-74	47
1390	$X_{8,9} \subset \mathbb{P}(1,2,3,3,4,5)$	I	$\frac{1}{5}, 12, 1389$	29	-54	36
1249	$X_{8,10} \subset \mathbb{P}(1,2,3,4,4,5)$	II_1	$\frac{1}{4}, 36, 1159$	30	-56	37
1179	$X_{9,10} \subset \mathbb{P}(1,2,3,4,5,5)$	I	$\frac{1}{5}, 15, 1162$	31	-58	37
1171	$X_{8,12} \subset \mathbb{P}(1,2,3,4,5,6)$	II_1	$\frac{1}{5}, 30, 1159$	36	-68	43
1165	$X_{10,11} \subset \mathbb{P}(1,2,3,4,5,7)$	I	$\frac{1}{7}, 11, 1162$	35	-66	41
1164	$X_{9,12} \subset \mathbb{P}(1,2,3,4,5,7)$	I	$\frac{1}{5}, 18, 1160;\ \frac{1}{7}, 9, 1162$	37	-70	43
1163	$X_{10,12} \subset \mathbb{P}(1,2,3,4,5,8)$	I	$\frac{1}{8}, 8, 1162$	38	-72	44
1161	$X_{12,14} \subset \mathbb{P}(1,2,3,4,7,10)$	I	$\frac{1}{10}, 8, 1160$	47	-90	53
1156	$X_{10,12} \subset \mathbb{P}(1,2,3,5,5,7)$	I	$\frac{1}{5}, 20, 1149;\ \frac{1}{7}, 12, 1155$	37	-70	42
1154	$X_{10,14} \subset \mathbb{P}(1,2,3,5,7,7)$	I	$\frac{1}{7}, 14, 1149$	43	-82	48

Continued on next page

1152	$X_{10,15} \subset \mathbb{P}(1,2,3,5,7,8)$	I	$\frac{1}{7}, 15, 1147;\ \frac{1}{8}, 10, 1149$	47	-90	52
1151	$X_{12,14} \subset \mathbb{P}(1,2,3,5,7,9)$	I	$\frac{1}{5}, 28, 1144;\ \frac{1}{9}, 12, 1149$	45	-86	50
1150	$X_{14,15} \subset \mathbb{P}(1,2,3,5,7,12)$	I	$\frac{1}{12}, 6, 1149$	51	-98	56
1148	$X_{15,16} \subset \mathbb{P}(1,2,3,5,8,13)$	I	$\frac{1}{13}, 6, 1147$	56	-108	61
1145	$X_{14,18} \subset \mathbb{P}(1,2,3,7,9,11)$	I	$\frac{1}{11}, 14, 1144$	59	-114	64
1121	$X_{10,12} \subset \mathbb{P}(1,2,4,5,5,6)$	II_1	$\frac{1}{5}, 40, 1112$	33	-62	39
1114	$X_{10,14} \subset \mathbb{P}(1,2,4,5,6,7)$	II_1	$\frac{1}{6}, 35, 1112$	38	-72	44
1083	$X_{12,16} \subset \mathbb{P}(1,2,5,6,7,8)$	II_1	$\frac{1}{5}, 48, 1067;\ \frac{1}{7}, 40, 1078$	41	-78	46
1080	$X_{14,15} \subset \mathbb{P}(1,2,5,6,7,9)$	I	$\frac{1}{9}, 15, 1079$	41	-78	45
1077	$X_{18,22} \subset \mathbb{P}(1,2,5,9,11,13)$	I	$\frac{1}{13}, 18, 1076$	60	-116	63
1068	$X_{14,18} \subset \mathbb{P}(1,2,6,7,8,9)$	II_1	$\frac{1}{8}, 45, 1067$	44	-84	49
867	$X_{10,12} \subset \mathbb{P}(1,3,3,4,5,7)$	I	$\frac{1}{7}, 10, 866$	31	-58	36
640	$X_{10,12} \subset \mathbb{P}(1,3,4,4,5,6)$	T^1		28	-52	33
547	$X_{12,13} \subset \mathbb{P}(1,3,4,5,6,7)$	I	$\frac{1}{7}, 13, 545$	30	-56	34
546	$X_{12,15} \subset \mathbb{P}(1,3,4,5,6,9)$	I	$\frac{1}{9}, 9, 545$	34	-64	38
544	$X_{12,14} \subset \mathbb{P}(1,3,4,5,7,7)$	I	$\frac{1}{7}, 14, 539$	32	-60	35
542	$X_{12,15} \subset \mathbb{P}(1,3,4,5,7,8)$	I	$\frac{1}{7}, 15, 537;\ \frac{1}{8}, 12, 539$	34	-64	37
541	$X_{14,15} \subset \mathbb{P}(1,3,4,5,7,10)$	I	$\frac{1}{10}, 10, 539$	36	-68	39
540	$X_{14,16} \subset \mathbb{P}(1,3,4,5,7,11)$	I	$\frac{1}{11}, 8, 539$	38	-72	41
538	$X_{15,16} \subset \mathbb{P}(1,3,4,5,8,11)$	I	$\frac{1}{11}, 10, 537$	39	-74	42
535	$X_{20,21} \subset \mathbb{P}(1,3,4,7,10,17)$	I	$\frac{1}{17}, 6, 534$	52	-100	54
509	$X_{14,15} \subset \mathbb{P}(1,3,5,6,7,8)$	I	$\frac{1}{8}, 14, 508$	32	-60	35
453	$X_{12,14} \subset \mathbb{P}(1,4,4,5,6,7)$	T^1		28	-52	32
359	$X_{14,16} \subset \mathbb{P}(1,4,5,6,7,8)$	T^1		29	-54	32
358	$X_{12,20} \subset \mathbb{P}(1,4,5,6,7,10)$	II_1	$\frac{1}{7}, 27, 355$	36	-68	39
357	$X_{18,20} \subset \mathbb{P}(1,4,5,6,9,14)$	I	$\frac{1}{14}, 8, 356$	38	-72	40
354	$X_{18,20} \subset \mathbb{P}(1,4,5,7,9,13)$	I	$\frac{1}{13}, 10, 353$	37	-70	38
338	$X_{16,18} \subset \mathbb{P}(1,4,6,7,8,9)$	T^1		30	-56	33
297	$X_{18,20} \subset \mathbb{P}(1,5,6,7,9,11)$	I	$\frac{1}{11}, 12, 296$	31	-58	32
279	$X_{18,30} \subset \mathbb{P}(1,6,8,9,10,15)$	T^1		36	-68	38
265	$X_{24,30} \subset \mathbb{P}(1,8,9,10,12,15)$	T^1		30	-56	31
37	$X_{12,14} \subset \mathbb{P}(2,3,4,5,6,7)$	T^1		18	-32	23

Table 4.3: Codimension 3: $h^{1,1}(X) = 1$ and $h^0(X, T_X) = 0$ in all cases.

grdb	variety	method	$\frac{1}{r}$, #nodes, target id	$h^{2,1}$	$e(X)$	$h^1(T_X)$
26988	$X_{2,2\dots} = X_{2,2,2} \subset \mathbb{P}^6$	$c_3(T_X)$		14	−24	27
24077	$X_{2,3\dots} \subset \mathbb{P}(1,1,1,1,1,1,2)$	$\mathrm{I} - T^1$	$\frac{1}{2}, 7, 24076$	14	−24	27
20543	$X_{3,3\dots} \subset \mathbb{P}(1,1,1,1,1,2,2)$	I − I	$\frac{1}{2}, 8, 20522$	15	−26	28
20523	$X_{3,3\dots} \subset \mathbb{P}(1,1,1,1,1,2,3)$	I − I	$\frac{1}{3}, 6, 20522$	17	−30	30
16338	$X_{3,3\dots} \subset \mathbb{P}(1,1,1,1,2,2,2)$	I − I	$\frac{1}{2}, 10, 16225$	18	−32	31
16226	$X_{3,4\dots} \subset \mathbb{P}(1,1,1,1,2,2,3)$	I − I	$\frac{1}{2}, 11, 16204$; $\frac{1}{3}, 7, 16225$	21	−38	34
16205	$X_{4,4\dots} \subset \mathbb{P}(1,1,1,1,2,3,4)$	I − I	$\frac{1}{4}, 7, 16204$	25	−46	38
12062	$X_{4,4\dots} \subset \mathbb{P}(1,1,1,2,2,2,2)$	I − I	$\frac{1}{2}, 12, 11435$	15	−26	27
11436	$X_{4,4\dots} \subset \mathbb{P}(1,1,1,2,2,2,3)$	I − I	$\frac{1}{2}, 14, 11102$; $\frac{1}{3}, 8, 11435$	19	−34	31
11122	$X_{4,4\dots} \subset \mathbb{P}(1,1,1,2,2,3,3)$	I − I	$\frac{1}{2}, 17, 11002$; $\frac{1}{3}, 9, 11102$	24	−44	36
11105	$X_{4,5\dots} \subset \mathbb{P}(1,1,1,2,2,3,4)$	I − I	$\frac{1}{2}, 18, 10983$; $\frac{1}{4}, 8, 11102$	25	−46	37
11103	$X_{4,5\dots} \subset \mathbb{P}(1,1,1,2,2,3,5)$	I − I	$\frac{1}{2}, 19, 10982$; $\frac{1}{5}, 5, 11102$	28	−52	40
11003	$X_{4,5\dots} \subset \mathbb{P}(1,1,1,2,3,3,4)$	I − I	$\frac{1}{3}, 11, 10983$; $\frac{1}{4}, 9, 11002$	32	−60	44
10984	$X_{5,6\dots} \subset \mathbb{P}(1,1,1,2,3,4,5)$	I − I	$\frac{1}{2}, 27, 10961$; $\frac{1}{5}, 9, 10983$	34	−64	46
10962	$X_{6,7\dots} \subset \mathbb{P}(1,1,1,3,4,5,6)$	I − I	$\frac{1}{6}, 11, 10961$	50	−96	62
6859	$X_{4,5\dots} \subset \mathbb{P}(1,1,2,2,2,3,3)$	$\mathrm{I} - \mathrm{II}_1$	$\frac{1}{3}, 11, 6858$	21	−38	32
5962	$X_{5,5\dots} \subset \mathbb{P}(1,1,2,2,3,3,3)$	I − I	$\frac{1}{3}, 12, 5857$	20	−36	30
5865	$X_{5,6\dots} \subset \mathbb{P}(1,1,2,2,3,3,4)$	I − I	$\frac{1}{3}, 13, 5843$; $\frac{1}{4}, 10, 5857$	22	−40	32
5858	$X_{5,6\dots} \subset \mathbb{P}(1,1,2,2,3,3,5)$	I − I	$\frac{1}{3}, 14, 5839$; $\frac{1}{5}, 6, 5857$	26	−48	36
5844	$X_{6,6\dots} \subset \mathbb{P}(1,1,2,2,3,4,5)$	I − I	$\frac{1}{5}, 10, 5843$	25	−46	35
5840	$X_{6,7\dots} \subset \mathbb{P}(1,1,2,2,3,5,7)$	I − I	$\frac{1}{7}, 6, 5839$	34	−64	44
5515	$X_{6,6\dots} \subset \mathbb{P}(1,1,2,3,3,3,4)$	I − I	$\frac{1}{3}, 15, 5261$; $\frac{1}{4}, 11, 5514$	22	−40	31
5302	$X_{6,6\dots} \subset \mathbb{P}(1,1,2,3,3,4,4)$	I − I	$\frac{1}{3}, 17, 5200$; $\frac{1}{4}, 12, 5261$	25	−46	34
5267	$X_{6,7\dots} \subset \mathbb{P}(1,1,2,3,3,4,5)$	I − I	$\frac{1}{3}, 18, 5161$; $\frac{1}{5}, 11, 5261$	26	−48	35
5264	$X_{6,6\dots} \subset \mathbb{P}(1,1,2,3,3,4,5)$	I − I	$\frac{1}{3}, 19, 5159$; $\frac{1}{4}, 13, 5258$; $\frac{1}{5}, 7, 5261$	30	−56	39
5262	$X_{6,7\dots} \subset \mathbb{P}(1,1,2,3,3,4,7)$	I − I	$\frac{1}{3}, 20, 5158$; $\frac{1}{7}, 5, 5261$	32	−60	41
5259	$X_{6,8\dots} \subset \mathbb{P}(1,1,2,3,3,5,8)$	I − I	$\frac{1}{3}, 23, 5154$; $\frac{1}{8}, 5, 5258$	38	−72	47
5201	$X_{6,7\dots} \subset \mathbb{P}(1,1,2,3,4,4,5)$	I − I	$\frac{1}{4}, 14, 5161$; $\frac{1}{5}, 12, 5200$	30	−56	39
5175	$X_{6,7\dots} \subset \mathbb{P}(1,1,2,3,4,5,5)$	I − I	$\frac{1}{5}, 13, 5159$; $\frac{1}{5}, 8, 5161$	36	−68	45
5162	$X_{7,8\dots} \subset \mathbb{P}(1,1,2,3,4,5,6)$	I − I	$\frac{1}{3}, 24, 5138$; $\frac{1}{6}, 12, 5161$	32	−60	41
5160	$X_{6,8\dots} \subset \mathbb{P}(1,1,2,3,4,5,7)$	I − I	$\frac{1}{4}, 17, 5155$; $\frac{1}{7}, 7, 5159$	42	−80	51
5139	$X_{8,9\dots} \subset \mathbb{P}(1,1,2,4,5,6,7)$	I − I	$\frac{1}{7}, 14, 5138$	42	−80	51
4999	$X_{8,8\dots} \subset \mathbb{P}(1,1,3,4,4,5,5)$	I − I	$\frac{1}{4}, 19, 4936$; $\frac{1}{5}, 15, 4985$	29	−54	36
4988	$X_{8,9\dots} \subset \mathbb{P}(1,1,3,4,4,5,6)$	I − I	$\frac{1}{4}, 20, 4912$; $\frac{1}{6}, 14, 4985$	30	−56	37
4986	$X_{8,9\dots} \subset \mathbb{P}(1,1,3,4,4,5,9)$	I − I	$\frac{1}{4}, 23, 4910$; $\frac{1}{9}, 5, 4985$	39	−74	46
4937	$X_{8,9\dots} \subset \mathbb{P}(1,1,3,4,5,5,6)$	I − I	$\frac{1}{5}, 17, 4912$; $\frac{1}{6}, 15, 4936$	33	−62	40
4914	$X_{9,10\dots} \subset \mathbb{P}(1,1,3,4,5,6,7)$	I − I	$\frac{1}{4}, 25, 4894$; $\frac{1}{7}, 15, 4912$	35	−66	42
4913	$X_{8,9\dots} \subset \mathbb{P}(1,1,3,4,5,6,7)$	I − I	$\frac{1}{6}, 17, 4911$; $\frac{1}{7}, 7, 4912$	43	−82	50
4895	$X_{10,11\dots} \subset \mathbb{P}(1,1,3,5,6,7,8)$	I − I	$\frac{1}{8}, 17, 4894$	43	−82	50
4849	$X_{10,11\dots} \subset \mathbb{P}(1,1,4,5,6,6,7)$	I − I	$\frac{1}{6}, 20, 4837$; $\frac{1}{7}, 18, 4848$	37	−70	43

Continued on next page

Table 4.3 continued from previous page

4838	$X_{11,12...} \subset \mathbb{P}(1,1,4,5,6,7,8)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{5}, 27, 4823;\ \frac{1}{8}, 18, 4837$	39	-74	45
4824	$X_{12,13...} \subset \mathbb{P}(1,1,4,6,7,8,9)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{9}, 20, 4823$	46	-88	52
4809	$X_{14,15...} \subset \mathbb{P}(1,1,5,7,8,9,10)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{10}, 23, 4808$	50	-96	55
4796	$X_{16,17...} \subset \mathbb{P}(1,1,6,8,9,10,11)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{11}, 26, 4795$	54	-104	59
2420	$X_{6,7...} \subset \mathbb{P}(1,2,2,3,3,4,5)$	$\mathrm{I}-\mathrm{II}_1$	$\frac{1}{5}, 8, 2419$	21	-38	29
2404	$X_{9,10...} \subset \mathbb{P}(1,2,2,3,5,7,9)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{9}, 8, 2403$	32	-60	39
1409	$X_{7,8...} \subset \mathbb{P}(1,2,3,3,4,4,5)$	II_1	$\frac{1}{4}, 21, 1389$	20	-36	27
1396	$X_{8,8...} \subset \mathbb{P}(1,2,3,3,4,5,5)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{5}, 10, 1390$	20	-36	26
1394	$X_{8,9...} \subset \mathbb{P}(1,2,3,3,4,5,7)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{7}, 8, 1390$	22	-40	28
1391	$X_{8,9...} \subset \mathbb{P}(1,2,3,3,4,5,8)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{8}, 6, 1390$	24	-44	30
1252	$X_{8,9...} \subset \mathbb{P}(1,2,3,4,4,5,5)$	$\mathrm{I}-\mathrm{II}_1$	$\frac{1}{5}, 11, 1249$	20	-36	26
1250	$X_{8,9...} \subset \mathbb{P}(1,2,3,4,4,5,7)$	$\mathrm{I}-\mathrm{II}_1$	$\frac{1}{7}, 7, 1249$	24	-44	30
1184	$X_{8,9...} \subset \mathbb{P}(1,2,3,4,5,5,6)$	$\mathrm{I}-\mathrm{II}_1$	$\frac{1}{5}, 13, 1171$	24	-44	30
1180	$X_{9,10...} \subset \mathbb{P}(1,2,3,4,5,5,7)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{5}, 13, 1165;\ \frac{1}{7}, 9, 1179$	23	-42	28
1168	$X_{9,10...} \subset \mathbb{P}(1,2,3,4,5,7,7)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{7}, 10, 1164;\ \frac{1}{7}, 8, 1165$	28	-52	33
1166	$X_{10,11...} \subset \mathbb{P}(1,2,3,4,5,7,9)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{9}, 9, 1165$	27	-50	32
1157	$X_{10,12...} \subset \mathbb{P}(1,2,3,5,5,7,12)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{5}, 19, 1150;\ \frac{1}{12}, 5, 1156$	33	-62	37
1153	$X_{10,12...} \subset \mathbb{P}(1,2,3,5,7,8,9)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{8}, 9, 1151;\ \frac{1}{9}, 11, 1152$	37	-70	41
1090	$X_{12,13...} \subset \mathbb{P}(1,2,5,6,7,7,8)$	$\mathrm{I}-\mathrm{II}_1$	$\frac{1}{7}, 15, 1083$	27	-50	31
1081	$X_{14,15...} \subset \mathbb{P}(1,2,5,6,7,9,11)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{11}, 12, 1080$	30	-56	33
868	$X_{10,12...} \subset \mathbb{P}(1,3,3,4,5,7,10)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{10}, 7, 867$	25	-46	29
641	$X_{10,11...} \subset \mathbb{P}(1,3,4,4,5,6,7)$	$\mathrm{I}-T^1$	$\frac{1}{7}, 9, 640$	20	-36	24
568	$X_{10,11...} \subset \mathbb{P}(1,3,4,5,5,6,7)$	II_1	$\frac{1}{5}, 22, 545$	21	-38	25
548	$X_{12,13...} \subset \mathbb{P}(1,3,4,5,6,7,10)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{10}, 8, 547$	23	-42	26
543	$X_{12,14...} \subset \mathbb{P}(1,3,4,5,7,8,11)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{8}, 11, 540;\ \frac{1}{11}, 7, 542$	28	-52	30
510	$X_{14,15...} \subset \mathbb{P}(1,3,5,6,7,8,11)$	$\mathrm{I}-\mathrm{I}$	$\frac{1}{11}, 9, 509$	24	-44	26
454	$X_{12,13...} \subset \mathbb{P}(1,4,4,5,6,7,9)$	$\mathrm{I}-T^1$	$\frac{1}{9}, 8, 453$	21	-38	24
392	$X_{12,13...} \subset \mathbb{P}(1,4,5,5,6,7,8)$	T^1		20	-36	23
326	$X_{14,15...} \subset \mathbb{P}(1,5,5,6,7,8,9)$	T^1		20	-36	22
298	$X_{16,17...} \subset \mathbb{P}(1,5,6,7,8,9,10)$	T^1		20	-36	22

Part II

The homogeneous land

Chapter 5

Preliminaries, part II

We move our analysis to subvarieties of the Grassmannian. Our final purpose for the second part of this Thesis is to provide a new version of the Griffiths's theorem for hypersurfaces and complete intersections in Grassmannian varieties $\mathrm{Gr}(k,n)$. In particular we are interested in a constructive result, like the already existings one for hypersurfaces and complete intersections in projective space. In order to reach enough familiarity with the subject, we spend quite a lot of time playing with new construction of subvarieties of $\mathrm{Gr}(k,n)$, with the main focus surfaces of general type. We recap here some instruments and few results in the literature on the topic. We want to stress the fact that many of these constructions actually holds - with minor modifications - in the more general context of homogeneous varieties; however in this thesis we decided to keep the focus only on the Grassmannian, leaving extensions for future works.

5.1 How to effectively use representation theory

A fundamental tool for all the computations in the Grassmannian (or more in general, in a homogeneous space) is Bott's theorem. We do not want to indulge in a deep theoretical introduction to the subject: we are actually interested in just a tour of its effective power in computations. The interested reader can find a comprehensive introduction in Jerzy Weyman's book, [118]. We recall here an excellent synthesis for non-experts, that can be found in [38]. A weight of the maximal torus of diagonal matrices in GL_n is an n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$. It is *dominant* if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We will often use the shorthand $\lambda = (n_1^{a_1}, \ldots, n_k^{a_k})$ meaning n_i is repeated a_i times in the tuple. If λ is a dominant weight with $\lambda_n \geq 0$ then λ yields a partition of $m = \sum \lambda_i$ and we denote this $\lambda \vdash m$. If it is clear that λ is a partition then we do not include the trailing zeros in the tuple.

Given an n-dimensional vector space V the irreducible representations of $\mathrm{GL}_n \simeq \mathrm{GL}(V)$ are in one-to-one correspondence with the dominant weights. We write $\mathbb{S}_\lambda V$ for the corresponding Schur functor, i.e. the irreducible representation associated to λ. We will write that $\lambda \vdash n$ to say that λ is a partition of n (or any other number involved). We have $\mathbb{S}_{(1^r)}V = \wedge^r V$, $\mathbb{S}_\lambda V \otimes \wedge^n V = \mathbb{S}_{\lambda+(1^n)}$ and $\mathbb{S}_\lambda V^* = \mathbb{S}_{(-\lambda_n,\dots,-\lambda_1)}V$. If V and W are vector spaces we have the Cauchy formula for $\mathrm{Sym}^k(V \otimes W)$ as $\mathrm{GL}(V) \times \mathrm{GL}(W)$-representation, namely

$$\mathrm{Sym}^k(V \otimes W) = \bigoplus_{\lambda \vdash k} \mathbb{S}_\lambda V \otimes \mathbb{S}_\lambda W.$$

If λ is a partition, denote by λ' the partition obtained by transposing the corresponding Young diagram. For example if $\lambda = (5,2,1)$, then $\lambda' = (3,2,1,1,1)$. The formula for the exterior power of the product $V \otimes W$ is then

$$\bigwedge^k (V \otimes W) = \bigoplus_{\lambda \vdash k} \mathbb{S}_\lambda V \otimes \mathbb{S}_{\lambda'} W. \tag{5.1}$$

Many other plethysm formulae can be computed: we list here the formulae we will use throughout.

$$\mathrm{Sym}^k(\bigwedge^2 V) = \bigoplus_{\lambda \vdash 2k, \lambda_i' \text{ even}} \mathbb{S}_\lambda V; \tag{5.2}$$

$$\mathrm{Sym}^k(\mathrm{Sym}^2 V) = \bigoplus_{\lambda \vdash 2k, \lambda_i \text{ even}} \mathbb{S}_\lambda V; \tag{5.3}$$

$$\bigwedge^k(\mathrm{Sym}^2 V) = \bigoplus_{\lambda} \mathbb{S}_{2[\lambda]} V. \tag{5.4}$$

The last formula requires a bit of explanation. Given a partition λ of k with distinct parts, let $2[\lambda]$ denote the partition of $2k$ whose main-diagonal hook lengths are $2\lambda_1, \dots, 2\lambda_k$, and whose i-th part has length $\lambda_i + 1$. The sum is over all partitions λ with distinct parts such that λ has at most n parts.
The Littlewood-Richardson formula will be important as well. It computes the tensor product of two Schur functor as

$$\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V = \bigoplus_{\nu \vdash \lambda+\mu} c^\nu_{\lambda,\mu} \mathbb{S}_\nu V. \tag{5.5}$$

The coefficients $c^{\nu}_{\lambda,\mu}$ can be computed in a combinatorial way. We used the "SchurRings" package of Macaulay2 or the "lrcalc" package of Sage.
Finally, let $\mathbb{S}_\lambda V$ an irreducible $\mathrm{GL}(V)$-module with highest weight $(\lambda_1, \dots, \lambda_n)$ and $\lambda_1 \geq \dots \geq \lambda_n$. The dimension of $\mathbb{S}_\lambda V$ is given by the Weyl character formula,

$$\dim(\mathbb{S}_\lambda V) = \prod_{1\leq i<j\leq n} \frac{\lambda_i - \lambda_j + j - i}{j-i} \tag{5.6}$$

5.1.1 Bott's theorem for the Grassmannian

Let $\mathrm{Gr}(k,n)$ be the Grassmannian of k-dimensional subspaces of V_n and let

$$0 \to \mathcal{S} \to \mathcal{O}_G \otimes V \to \mathcal{Q} \to 0$$

be the tautological sequence on $\mathrm{Gr}(k,n)$. By functoriality the Schur functors may be applied to vector bundles on the $\mathrm{Gr}(k,n)$, in particular to the tautological sub and quotient bundles $\mathcal{S}$ and $\mathcal{Q}$.
Consider two dominant weights $\alpha = (\alpha_1, \dots, \alpha_{n-r})$ and $\beta = (\beta_1, \dots, \beta_r)$ and their concatenation $\gamma = (\gamma_1, \dots, \gamma_r)$. Let $\delta = (n-1, \dots, 0)$ and consider $\gamma + \delta$. Write $\mathrm{sort}(\gamma+\delta)$ for the sequence obtained by arranging the entries of $\gamma+\delta$ in non-increasing order, and define $\tilde{\gamma} = \mathrm{sort}(\gamma+\delta) - \delta$.

Theorem 5.1.1 (Bott, cf. [38])**.** *With the above notation, if $\gamma+\delta$ has repeated entries, then*

$$H^i(\mathrm{Gr}(k,n), \mathbb{S}_\alpha \mathcal{Q} \otimes \mathbb{S}_\beta \mathcal{S}) = 0$$

for all $i \geq 0$. Otherwise, writing l for the number of pairs (i,j) with $1 \leq i < j \leq n$ and $\gamma_i - i < \gamma_j - j$, we have

$$H^l(\mathrm{Gr}(k,n), \mathbb{S}_\alpha \mathcal{Q} \otimes \mathbb{S}_\beta \mathcal{S}) = \mathbb{S}_{\tilde{\gamma}} V$$

and $H^i(\mathrm{Gr}(k,n), \mathbb{S}_\alpha \mathcal{Q} \otimes \mathbb{S}_\beta \mathcal{S}) = 0$ for $i \neq l$.

5.1.2 A worked example: the $\mathrm{Gr}(2,5)$ case

We want to give an explicit example of the above technique. We will refer in particular to the $\mathrm{Gr}(2,5)$ case, since it will be relevant for the next chapters. We calculate here the dimension of the various cohomology groups $H^i(\mathrm{Gr}(2,5), \Omega^2_G(k))$. The cotangent bundle Ω^1_G is isomorphic to $S \otimes \mathcal{Q}^*$; its second exterior power Ω^2 is not irreducible, and decomposes as

$$\Omega^2_{\mathrm{Gr}(2,5)} = \mathbb{S}_{(0,0,-2)}\mathcal{Q} \otimes \mathbb{S}_{(1,1)}\mathcal{S} \oplus \mathbb{S}_{(0,-1,-1)}\mathcal{Q} \otimes \mathbb{S}_{(2,0)}\mathcal{S}.$$

Tensoring with $\mathcal{O}_G(k) = (\bigwedge^5 \mathcal{Q})^{\otimes k}$ gives

$$\Omega^2_{\mathrm{Gr}(2,5)}(k) = \mathbb{S}_{(k,k,k-2)}\mathcal{Q} \otimes \mathbb{S}_{(1,1)}\mathcal{S} \oplus \mathbb{S}_{(k,k-1,k-1)}\mathcal{Q} \otimes \mathbb{S}_{(2,0)}\mathcal{S}.$$

Consider the two concatenations $\gamma = (k, k, k-2, 1, 1)$ and $\gamma' = (k, k-1, k-1, 2, 0)$ and the vector $\delta = (4, 3, 2, 1, 0)$. We have

$$\gamma + \delta = (k+4, k+3, k, 2, 1),\ \gamma' + \delta = (k+4, k+2, k+1, 3, 0).$$

For $k > 1$ and $k < -3$ both sequences do not have repeated terms. This means

$$H^i(\Omega^2_G(k)) = 0, k \geq 2, i \neq 0,\ H^i(\Omega^2_G(k)) = 0, k \leq -4, i \neq 6.$$

What about $-3 \leq k \leq 1$? For $k = 0$ it follows from the description of the cohomology of the Grassmannian that $H^i(\Omega^2_G) = 0$ for $i \neq 2$. For $k = 1, -1, -2$ both $\gamma + \delta$ and $\gamma + \delta'$ have repeated factors, and therefore all their cohomology groups vanish. For $k = -3$ the vector $\gamma' + \delta'$ does not have a repeated factor and on $\gamma_x - x = (-4, -6, -7, -2, -5)$ the number of negative differences is 5. We have $\widetilde{\gamma'} = \mathrm{sort}(\gamma' + \delta) - \delta = (-1, -2, -2, -2, -2)$. Therefore $H^i(\Omega^2_G(-3)) = 0$ for $i \neq 5$ and

$$H^5(\Omega^2_G(-3)) = V_{(-1,-2,-2,-2,-2)}.$$

By Weyl's formula it follows

$$\dim V_{(-1,-2,-2,-2,-2)} = \prod \frac{2 \cdot 3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4} = 5.$$

From similar computations one has in general

$$H^i(\mathrm{Gr}(2,5), \Omega^j(k)) = 0,\ 1 \leq i, j \leq 5, k \neq 0, (i, j, k) \neq (2, 5, -3) \neq (3, 5, -2) \neq (4, 1, 3).$$

We will use Bott theorem repeatedly throughout all the rest of the thesis.

5.2 Subvarieties in of the Grassmannian: state of the art

One of the most important tool for explicit constructing new varieties comes from the theory of vector bundles. Let $\mathcal{F}$ a globally generated vector bundle of rank c on a smooth variety Y and $s \in H^0(Y, \mathcal{F})$ a general global section. Then X, the scheme of zeroes of s is called complete intersection with respect to $\mathcal{F}$. X is smooth of dimension $\dim(Y) - c$ if it is not empty. The Koszul complex associated to s gives a resolution of $\mathcal{O}_X$:

$$0 \to \bigwedge^{r} \mathcal{F}^* \to \bigwedge^{r-1} \mathcal{F}^* \to \ldots \to \mathcal{F}^* \to \mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

The adjunction formula holds: we have

$$K_X = (K_Y + \det(\mathcal{F}))|_X.$$

If $\mathcal{E}$ is another vector bundle on Y, there exists a spectral sequence such that

$$\mathbf{E}_1^{-q,p} = H^p(Y, \mathcal{E} \otimes \bigwedge^{q} \mathcal{F}^*) \Rightarrow H^{p-q}(Y, \mathcal{E}|_X),$$

that in turn allows us to perform explicit computations. We are of course interested to the case $Y = \mathrm{Gr}(k, n)$ and $\mathcal{F}$ an homogeneous vector bundle. A vector bundle $\mathcal{E}$ on a space of type $Y = \mathcal{G}/P$ with $\mathcal{G}$ a connected, simply connected, semisimple complex Lie group and P parabolic is *homogeneous* if there is an algebraic representation E of P such that $\mathcal{E}$ is isomorphic to the twisted product $\mathcal{G} \times_P E$ which is by definition the quotient of $\mathcal{G} \times E$ by the action $p \cdot (g, z) = (gp^{-1}, p \cdot z)$. Although we appreciate the generality of this definition, we prefer a more operative one, especially in our context. In particular, if $Y = \mathrm{Gr}(k, n)$, an irreducible vector bundle $\mathcal{F}$ will be

$$\mathcal{F} = \bigoplus \mathbb{S}_\lambda \mathcal{S} \otimes \mathbb{S}_\mu \mathcal{Q} \otimes \mathcal{O}(k),$$

where $\mathcal{S}$ and $\mathcal{Q}$ are (resp.) the tautological rank k bundle and the rank $n-k$ quotient bundle. Such a bundle will be generated by global sections if and only if the highest weight $(\lambda + k, \mu)$ is dominant. Notice that all complete intersections in Grassmannian are indeed a particular case of this construction.

The classification of varieties arising in this way is one of the most active topics of research. Indeed, it is important for classifications of Fano in high dimension, hyperkähler geometry and derived categories. Still, we only have few partial results. The first and most classical in this sense is the Küchle list ([79]) of 17 families of Fano fourfolds of index 1. The geometry of these varieties is still object of research, as in the recent series of works by Kuznetsov ([82], [80]). Of particular interests are indeed the Fano fourfolds in this list whose middle Hodge structure contains (at least numerically) a sub-Hodge structure of weight 2. A generalisation of these ideas is contained in the theory of *Fano manifold of Calabi-Yau type*, see [69], that actually serves as the main motivation for many of our constructions. Other classifications include Calabi-Yau threefolds ([71]) and fourfolds with trivial canonical bundle ([13]). We summarise in the following theorem the known results on the topic

Theorem 5.2.1 (cf. [79], [71], [84],[13]). *There existsts*

- 17 *families of Fano fourfolds of index 1;*
- 33 *families of Calabi-Yau threefolds;*
- 48 *families of fourfolds with* $K_X \cong \mathcal{O}_X$*;*

that can be obtained as zero set of a general section of an homogeneous vector bundle $\mathcal{F}$ *on a Grassmannian* $\mathrm{Gr}(k,n)$*.*

In the next section we will start from the above lists and try to produce new examples of varieties in low dimension by producing examples of invariant subfamilies with respect to small groups. Then we establish a version of Griffiths residue theorem for a special case, namely the complete intersection one. An infinitesimal Torelli for these type of varieties is already known: namely we have the following theorem by Konno

Theorem 5.2.2 (cf.[77]). *Let* $X = X_{d_1,\dots,d_c} \subset \mathrm{Gr}(k,n)$ *a smooth complex intersection of multi-degree* $d_1,\dots,d_c$*. Then the infinitesimal Torelli holds for* X *provided that either*

- *the canonical bundle* K_X *is non-negative, or*
- *all* $d_i \geq 2$*, except the case of* $X_2 \subset \mathrm{Gr}(2,6)$

Few cases are left, not to mention the global case, as many other Hodge-theoretic problems that our method could help to solve. We are particularly intrigued by the construction and classification of Fano manifold of Calabi-Yau types. In the final section we list some conjectures and related problems we plan to work on in the near future.

Chapter 6

Invariant families and surfaces of general type

In this chapter we show how to produce new examples of varieties in low dimension (namely, 2 and 3). Indeed many of the varieties mentioned in the previous chapter seems to have the right numerology to be invariant under finite groups. We provide a bunch of significative examples and explicitely construct some new varieties in low dimension.

6.1 A tower of varieties in Gr(2,6) and Gr(2,7)

The classification of surfaces of general type is one of the most active areas of algebraic geometry. Many examples are known, but a detailed classification is still lacking (and maybe even impossible to accomplish), and several hard problems are still open. Recall that a surface S is said to be minimal if does not contain any -1 curve. Every surface can be obtained by a minimal one (its "minimal model") after a finite sequence of blowing ups of smooth points; this model is moreover unique if the the Kodaira dimension satisfies $k(S) \geq 0$. In particular we can reduce the study of the birational class of a surface S to the study of its minimal model. To each minimal surface of general type we will associate a triple of numerical invariants, (p_g, q, K_S^2), where $p_g := h^0(S, K_S)$ and $q := h^1(S, \mathcal{O}_S)$. These indeed determine all other classical numerical invariants, such as $e_{\text{top}}(S) = 12\chi(\mathcal{O}_S) - K_S^2$ and $P_m(S) := h^0(S, mK_S) = \chi(\mathcal{O}_S) + \binom{m}{2} K_S^2$. For a recent survey on the surfaces of general type we refer to [10].

Two very simple ways to produce surfaces of general type are complete intersections of sufficiently high degree or product of curves with $g \geq 2$. These produces surfaces with either large p_g or q. This is a particular manifestation of more general phenomenon: producing examples of surfaces of general type with low p_g and q is indeed quite difficult, and a complete classification is beyond the current level of research. A useful tool to produce such examples consist in the identification of families of surfaces of general type whose general member S is invariant with respect to a finite group H, and taking the quotient S/H. The archetypal example is due to Godeaux, and is realised as the quotient $Y_5/\mathbb{Z}/5$, where $Y_5 \subset \mathbb{P}^3$ is a quintic surface in $\mathbb{P}^3$ on which the group $\mathbb{Z}/5$ acts freely. Surfaces with $p_g = q = 0, K_S^2 = 1$ are therefore called *(numerical) Godeaux surfaces*. Similarly one can construct explicit examples of a surface with $p_g = q = 0, K_S^2 = 2$ as quotient for a $\mathbb{Z}/7$ action. Indeed surfaces with these prescribed invariants are called *(numerical) Campedelli surfaces*. We will recall later in full details the latter construction.

Finding examples of such invariant subfamilies in $\mathbb{P}^n$ can be difficult. On the other hand the lists of Küchle and Inoue-Ito-Miura provides an excellent source of potential candidates.

The starting point is the analysis of two Fano fourfolds of index 1 in Grassmannians Gr(2,6) and Gr(2,7). These Fanos are constructed as (resp.) zero locus of a general section of the twisted quotient bundle and 6-codimensional general linear section. They appears in Küchle list as (b3) and (b7) and were shown to be projectively equivalent in a recent work of Manivel ([84]). From these one can get to the level of surfaces by simply picking two furher hyperplane sections. These are surfaces of general type with $p_g = 13, K^2 = 42$.

We explicitely show how to construct an action of the dihedral group D_7 of order 14 on these Fanos, and how to pick D_7-invariant linear subspaces such that the corresponding surfaces are smooth, with a free $\mathbb{Z}/7 \triangleleft D_7$ action. This in turn will allow us to produce new examples of surfaces of general type and Calabi-Yau threefolds.

Hodge numbers of linear sections of Gr(2,7) Let $G_7 = \mathrm{Gr}(2,7)$ and consider the following tower of linear sections

$$S_Z \subset W_Z \subset Z \subset G_7$$

where each member of the tower is given by the zero scheme of a general global section of $\mathcal{O}_{G_7}(1)^{\oplus r}$, $r = 6, 7, 8$. Equivalently, each of these is given by a general linear system $\Sigma \subset \bigwedge^2 V^*$ of the corresponding dimension, where we use $H^0(G_7, \mathcal{O}_{G_7}(1)) \cong \bigwedge^2 V_7^*$. Since $\omega_{G_7} \cong \mathcal{O}_{G_7}(-7)$ by adjunction is easy to see that Z is a prime Fano fourfold of index $\iota_Z = 1$, W_Z is a Calabi-Yau threefold (already famous in literature for its application in Mirror Symmetry, see [103]) and S_Z is a surface of general type with $\omega_{S_Z} = \mathcal{O}_{S_Z}(1)$. All of these three varieties shares $K^{\dim} = 42$. We can compute their Hodge numbers either using Koszul complex and Bott's theorem or using the tools from the previous chapters.

Hodge numbers of Z One has

$$\begin{array}{ccccccccc} 0 & & 6 & & 57 & & 6 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

with moreover $h^1(Z, T_Z) = 42$.

Hodge numbers of W_Z One has

$$\begin{array}{ccccccc} 1 & & 50 & & 50 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

with $h^1(W_Z, T_{W_Z}) = h^{2,1}(W_Z) = 50$.

Hodge numbers of S_Z One has

$$\begin{array}{ccccc} 13 & & 98 & & 13 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

with $h^1(S_Z, T_{S_Z}) = 56$.

Quotient bundle on Gr(2,6) Consider now the Grassmannian G_6=Gr(2,6) and $\mathcal{Q}(1)$ the rank four globally generated quotient bundle twisted by $\mathcal{O}_{G_6}(1)$. If λ is a general global section in $H^0(G_6, \mathcal{Q}(1))$ its zero locus Y_λ will be a smooth Fano fourfold, with

$$K_{Y_\lambda} = (K_{G_6} \otimes det(\mathcal{Q}(1))|_{Y_\lambda} = \mathcal{O}_{Y_\lambda}(-6+5) = \mathcal{O}_{Y_\lambda}(-1).$$

We have a concrete description of the space of the global section of $\mathcal{Q}(1)$ given in [84]. More precisely by [20] we have

$$H^0(G_6, \mathcal{Q}(1)) = \mathrm{Ker}(\lrcorner\colon (\bigwedge^2 V_6^*) \otimes V_6 \to V_6^*),$$

where $\lrcorner$ is the contraction operator.
In particular we have that any $\lambda \in H^0(G, \mathcal{Q}(1))$ is an element in $\mathrm{Hom}(\bigwedge^2 V_6, V_6)$. For every λ the corresponding Y_λ will be

$$Y_\lambda = \{< a, b > \in \mathrm{Gr}(2, 6) \mid \lambda(a, b) \in < a, b >\}. \tag{6.1}$$

By taking two further hyperplane sections one gets even here a tower

$$S_Y \subset W_Y \subset Y_\lambda \subset G_6.$$

As one can check the invariants of the towers are the same once fixed the dimension: the reason for this coincidence has been explained by Manivel in [84]

Theorem 6.1.1 ([84])**.** *Z and Y_λ are projectively equivalent.*

One has (see [71] for the Calabi-Yau, and easy to see by hand as in the surface case) that W_Z-W_Y and S_Z-S_Y shares the same invariants as well.
We now start by defining our quotient construction, working both with the Y and Z model. We will focus on the cases of main interest for us, these being the fourfolds Y, Z and the surfaces S_Y, S_Z, but of course everything can be adapted to the Calabi-Yau case W_Y, W_Z. Often, when computations will be identical, we will go into the details only for one model and just sketch the other. We will start defining in the following two different action of D_7, the dihedral group of order 14, on V_6 and V_7.

6.1.1 Two representations of D_7

D_7 acting on V_6

Consider now the group D_7, acting on $\mathbb{C}^6$ via

$$\tau_6 = \frac{1}{7}(1,2,3,4,5,6),\ \sigma_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

that is $\sigma_6(v_i) = v_{7-i}$. It is easy to see that τ_6 and σ_6 satisfies the relations of the dihedral group, that is

$$\tau_6^7 = \sigma_6^2 = \mathrm{Id},\ \sigma_6\tau_6^i = \tau_6^{7-i}\sigma_6.$$

The choice of this representation is motivated by some famous analogous constructions in the theory of surfaces of general type (for example the standard construction of a *Godeaux surface* as a $\mathbb{Z}/5$ quotient of a smooth quintic surface.
The action of σ_6 pass to $\bigwedge^2 V_6$ via $V_6^{\otimes 2}$, with the rule

$$\sigma_6(v_i \wedge v_j) = v_{7-i} \wedge v_{7-j},$$

and as well to $\mathrm{Gr}(2,6)$, that we can identify as the set of totally decomposable 2-skew tensors in $\mathbb{P}(\bigwedge^2 V_6)$. With a little abuse of notation we will denote with ρ_6 both this representation of $\mathbb{C}^6$ and on $\bigwedge^2 \mathbb{C}^6$

We want now to consider the subspace $\mathcal{Y}^{\rho_6}$ given by the D_7-invariant Y_λ under the given representation ρ_6, that is

$$\mathcal{Y}^{\rho_6} := \{Y_\lambda \mid \lambda(g \cdot [p]) \in Y_\lambda,\ g \in D_7,\ [p] \in Y_\lambda\},$$

where the D_7 action is computed according to ρ_6.

Proposition 6.1.2. *The family $\mathcal{Y}^{\rho_6}$ of D_7 invariant fourfolds of type Y_λ has general member*

$\lambda = v_1 \otimes (c_{2,6} v_2^* \wedge v_6^* + c_{3,5} v_3^* \wedge v_5^*) + v_2 \otimes (c_{3,6} v_3^* \wedge v_6^* + c_{4,5} v_4^* \wedge v_5^*) + v_3 \otimes (c_{1,2} v_1^* \wedge v_2^* + c_{4,6} v_4^* \wedge v_6^*) +$

$+v_4 \otimes (c_{4,6} v_1^* \wedge v_3^* + c_{1,2} v_5^* \wedge v_6^*) + v_5 \otimes (c_{3,6} v_1^* \wedge v_4^* + c_{4,5} v_2^* \wedge v_3^*) + v_6 \otimes (c_{2,6} v_1^* \wedge v_5^* + c_{3,5} v_2^* \wedge v_4^*)$

Proof. Let us start writing a general element in $\bigwedge^2 V_6^* \otimes V_6$: this will be

$$\lambda = \sum_{i,j,k} c_{i,j,k} v_i \otimes (v_j^* \wedge v_k^*).$$

If $a = \sum a_s v_s$ and $b = \sum b_s v_s$ are elements of V, we have that

$$\lambda(a, b) = \sum_i v_i (\sum_{j,k} c_{i,j,k} p_{j,k}),$$

where $p_{j,k} = a_j b_k - b_j a_k$. The action of D_7 in terms of the generators can be expressed as

$$\tau(\lambda(a, b)) = \sum \xi^i v_i (\sum_{j,k} c_{i,j,k} p_{j,k}),$$

$$\sigma(\lambda(a, b)) = \sum v_{7-i} (\sum_{j,k} c_{i,j,k} p_{j,k}).$$

On the other hand we have

$$\lambda(\tau(a, b)) = \sum v_i (\sum_{j,k} \xi^{j+k} c_{i,j,k} p_{j,k}),$$

$$\lambda(\sigma(a, b)) = \sum v_{7-i} (\sum_{j,k} c_{7-i,j,k} p_{7-j,7-k}).$$

This induces relations between $c_{i,j,k}$, namely

1. $c_{i,j,k} = c_{7-i,7-j,7-k}$;
2. $c_{i,j,k} = 0$ for $j + k \not\equiv i \pmod 7$

Expanding these conditions the statement follows. □

In order to get to a surface we need now to consider the zero set of a global section of $\mathcal{Q}(1) \oplus \mathcal{O}_G(1)^{\oplus 2}$: by Borel-Bott-Weil theorem we have

$$H^0(G, \mathcal{Q}(1) \oplus \mathcal{O}_G(1)^{\oplus 2}) = H^0(G, \mathcal{Q}(1)) \oplus (\bigwedge^2 V^*)^2,$$

therefore we want to realize S_{42} as $V(\lambda, h_1, h_2)$, whereas h_1, h_2 are two linear forms in Plücker coordinates. In order to preserve the surface we need to look for D_7 equivariant linear form as well: in particular, we need to work with the set

$$\mathcal{H}^{\rho_6} := \{(h_1, h_2) \in (\bigwedge^2 V^*)^2 \mid (h_1, h_2) \text{ preserved by } D_7 \text{ action } \}$$

These will come by three copies of the trivial induced representation: in coordinates we have to check that, if $p = \sum l_{i,j} v_i \wedge v_j$ and $h_i = \sum h^i_{i,j} v^*_i \wedge v^*_j$, then if $p \in V(h_1, h_2)$, then $g \cdot p \in V(h_1, h_2)$ as well. It is easy to see that the action of τ and σ combined implies that the two linear forms must be both of the form

$$h_i = h^i_{1,6}\ v^*_1 \wedge v^*_6 + h^i_{2,5}\ v^*_2 \wedge v^*_5 + h^i_{3,4}\ v^*_3 \wedge v^*_4.$$

Indeed we have

Proposition 6.1.3. *Any D_7 invariant surfaces $S^{\rho_6}_Z$ (with respect to the representation ρ_6) will be given by the triple $(\lambda_{\rho_6}, h_1, h_2)$, with λ_{ρ_6} as in proposition 6.1.2 above, and h_1, h_2 in $\mathcal{H}^{\rho_6}$.*

From Gr(2,6) to Gr(2,7) and D_7 action

In order to understand how the action of D_7 on V_7 works, we make explicit the identification between Y and Z. We use an alternative description given by Inoue-Ito-Miura, (cfr. [71], Proposition 4.1), that here we recall briefly. Suppose V is a linear space of dimension n, $\mathcal{E}$ a globally generated vector bundle on $Gr(k, V)$, s an element in $H^0(\mathcal{E}) \otimes (\bigwedge^k V)^*$ and $\bar{s}$ its image in $H^0(\mathcal{E}(1))$. We denote by $P_{\bar{s}}$ the linear section of $Gr(k, V \oplus \mathbb{C}) \subset \mathbb{P}(W \oplus \bigwedge^k W)$ given by the image of the map

$$\mathbb{P}(\bigwedge^k V) \hookrightarrow \mathbb{P}(H^0(\mathcal{E}) \oplus \bigwedge^k V); \quad [p] \to [\bar{s}(p), p],$$

where $W = V \oplus \mathbb{C}$.

One has that $\bar{s}$ is general if and only if $P_{\bar{s}}$ is, and $V(\bar{s})$ and $V(P_{\bar{s}})$ are projectively equivalent. This is exactly our case with $\mathcal{E} = \mathcal{Q}$ and $\bar{s} = \lambda$. Therefore computing the image of the map above one has that $Z = V(P_{\bar{s}}) \subset Gr(2, 7)$ is defined by the following 6 equations

$$Z = V(x_{1,7} - c_{2,6}x_{2,6} - c_{3,5}x_{3,5}, x_{1,6} - c_{3,6}x_{2,5} - c_{4,5}x_{3,4}, x_{1,5} - c_{4,6}x_{2,4} - c_{1,2}x_{6,7}, \tag{6.2}$$

$$x_{1,4} - c_{1,2}x_{2,3} - c_{4,6}x_{5,7}, x_{1,3} - c_{3,6}x_{4,7} - c_{4,5}x_{5,6}, x_{1,2} - c_{2,6}x_{3,7} - c_{3,5}x_{4,6}).$$

In particular for a generic set of $c_{i,j}$ any linear equation f_i defining Z above can be represented by an element $A_i \in (\bigwedge^2 V_7)^*$ and any A_i is thus an antisymmetric matrix of maximal rank. To have a concrete visualization of it, we will have that for example A_1 will be given by

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This suggests indeed how the D_7 action on V_7 should work. In particular we define $\tau_7 = \frac{1}{7}(0,1,2,3,4,5,6)$ and $\sigma_7(v_i) = v_{9-i}$. This passes to $\bigwedge^2 V_7$ via

$$\tau_7(v_i \wedge v_j) = \epsilon^{i+j-2} v_i \wedge v_j$$

and

$$\sigma_7(v_i \wedge v_j) = v_{9-j} \wedge v_{9-i}.$$

Note that in this way $\sigma_7(v_1 \wedge v_k) = -v_1 \wedge v_{9-k}$, and this explain the difference of sign when passing from Y to Z. We denote this representation by ρ_7. With computations totally similar to the case $n = 6$, one find after rescaling the first coefficient of every equation that the maximal invariant family is indeed what we already found above

Lemma 6.1.4. *The maximal family Z_{ρ_7} of invariant fourfold with the action above defined is the complete intersection defined by following*

$$Z_{\rho_7} = V(x_{1,7} - \mu_1 x_{2,6} - \mu_2 x_{3,5}, x_{1,6} - \mu_3 x_{2,5} - \mu_4 x_{3,4}, x_{1,5} - \mu_6 x_{2,4} - \mu_5 x_{6,7},$$

$$x_{1,4} - \mu_5 x_{2,3} - \mu_6 x_{5,7}, x_{1,3} - \mu_3 x_{4,7} - \mu_4 x_{5,6}, x_{1,2} - \mu_1 x_{3,7} - \mu_2 x_{4,6}).$$

Notice that in any of the above equations the sum $i+j \equiv k \pmod 7$ is constant k ranging from 1 to 7, with 2 missing.
Similary the maximal family S_{ρ_7} is obtained by adding two copies coming from the trivial representations, that is two (linearly independent) hyperplanes in the coordinates $x_{3,6}, x_{4,5}.x_{2,7}$.

We want to rewrite the generic member of the above family of surfaces in a much more neat style. Recall that taking the 4-Pfaffians of a generic skew 7×7 matrix of linear forms yields the Plücker equations of the Grassmannian $\mathrm{Gr}(2,7)$. We can write our invariant family in the format

$$M = \begin{pmatrix} \mu_1 x_{3,7}+\mu_2 x_{4,6} & \mu_3 x_{4,7}+\mu_4 x_{5,6} & \mu_5 x_{2,3}+\mu_6 x_{5,7} & \mu_6 x_{2,4}+\mu_5 x_{6,7} & \mu_3 x_{2,5}+\mu_4 x_{3,4} & \mu_1 x_{2,6}+\mu_2 x_{3,5} \\ & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & \epsilon_1 x_{4,5} \\ & & x_{3,4} & x_{3,5} & \epsilon_2 x_{4,5} & x_{3,7} \\ & & & x_{4,5} & x_{4,6} & x_{4,7} \\ & & & & x_{5,6} & x_{5,7} \\ & & & & & x_{6,7} \end{pmatrix}$$

where of $S_Z \subset \mathbb{P}^{12}$ is

$$S_Z = V(\mathrm{Pf}(4, M)). \tag{6.3}$$

The parameters ϵ_1 and ϵ_2 come from the solution of the system of two equations in the $x_{3,6}, x_{4,5}, x_{2,7}$. Equation for the generic Calabi-Yau and fourfold can be easily accessed plugging back in $x_{3,6}, x_{2,7}$.

6.1.2 Simultaneous smoothness and fix locus of the action

Before taking the quotient, we need to address the question of the smoothness of our specific fourfolds Y_λ and Z. As said before, by Inoue-Ito-Miura it suffices to check this for the Z-model (since the smoothness of Z implies the generality of λ, and therefore the smoothness of Y_λ).

Lemma 6.1.5. *The general surface S_Z constructed above is smooth.*

Proof. The smoothness of Z can be checked in several ways, for example by computing the infinitesimal deformation module of the affine cone of the general member or with a computation in local coordinates. We require our coefficients to be sufficiently general, for example all distinct numbers. On the other hand it is easy to to produce singular example with some special choice of coefficients. For example by picking all $\mu_i = 1$ one gets a nodal surface, though we have not been able to determine the degree of the singular locus yet. We propose here an alternative computer-free method coming from the theory of *exterior differential systems* (see [30]). We use a criterion for a point in a linear section of a Grassmannian of planes that is sufficient for smoothness, but not necessary.

In general, let V a vector space, $\Sigma \subset \bigwedge^2(V^*)$ a linear subspace and Z_Σ the corresponding subvariety of the Grassmannian. For any $w \in V$, consider the vector space $H(w)$ defined as

$$H(w) = \{a \in V \mid \Omega(a, w) = 0, \text{ for all } \Omega \in \Sigma\}.$$

We say that w is Σ-regular if the dimension of $H(w)$ is minimal among all $w \in V$ and that a 2-plane $P \in Z_\Sigma$ is Σ-ordinary if P contains a Σ-regular vector. The relevant result is that any ordinary plane is actually a smooth point of Z_Σ.
Let us now apply this method to our case. Let us do first the surface S_Z. Fix a $w = \sum w_i v_i$: $H(w)$ is then exactly the space of point u in $\mathbb{C}^7$ that satisfies the system of equations 6.2, with two more in the coordinates $x_{3,6}, x_{4,5}, x_{2,7}$. This amounts to solve the linear system

$$M \cdot U = 0,$$

where

$$M = (\mu_k w_i)_{k,i}, \quad U = (u_1, \dots, u_7)^T.$$

One checks that for general μ_k and w_i the matrix has maximal rank (that is, the dimension of $H(w)$ is constantly zero for general choices) and that any plane P in S_Z contains a general w. $\square$

By applying the same method one checks

Lemma 6.1.6. *The general fourfolds Z and Y_λ constructed above are smooth.*

Proof. The above method works perfectly for every $P \in Z$, except $p_{3,6}, p_{4,5}, p_{2,7}$ (recall that these three points do not belong to S_Z). In fact one checks that for any w in these three planes the corresponding $H(w)$ has dimension two, instead of the expected one. Still, not everything is lost. As said above, the method here is only sufficient, but far from necessary. A local computation on the Grassmannian (using for example the chart $p_{1,2} = 1$) shows that even these three points are smooth points of Z. $\square$

Fix locus of the action

The Z-model Once estabilished the smoothness of the fourfolds Y_λ and Z (and the same for the surface S_Z) of the maximal D_7 invariant families, we have to compute the fixed locus for the elements of the group. The Y-model is identical, therefore we will just sketch the computations.
We have the following

Lemma 6.1.7. *The fixed locus for the action of the group D_7 is*

- *on the surface S_Z and on the Calabi-Yau W_Z it consist of the reducible union $\bigsqcup_{i=1}^7 C_i$, where each C_i is the union of a plane conic and 10 extra (disjoint) nodes. Moreover all C_i are conjugates under the normal subgroup $\mathbb{Z}/7$;*

- *on the fourfold Z it has 3 extra fixed points .*

Proof. Consider first the cyclic action of τ_7 as $\frac{1}{7}(0,1,2,3,4,5,6)$ when induced. In particular it sends

$$\tau_7(\sum \lambda_{i,j} v_i \wedge v_j) \to \sum \epsilon^{i+j-2} \lambda_{i,j} v_i \wedge v_j.$$

The fixed locus on $\mathbb{P}^{20}$ is the union of seven $\mathbb{P}^2$, each one with coordinates $\{x_{i,j}\}_{i+j \equiv k \ (mod\ 7)}$. A computer check shows that the cyclic fixed locus lies away from W and S. We give in the following a computer-free proof.
We have two types of fixed points: the coordinate points $p_{i,j}$ and any other of the form

$$\lambda_{i,j} v_i \wedge v_j, \text{ with } i+j \equiv \text{const } (mod\ 7). \tag{6.4}$$

It is easy to check that no coordinate points $p_{i,j}$ belongs to Z except $p_{3,6}, p_{2,7}, p_{4,5}$ (and they do not belong to S_Z). We claim now that any of the point of the second type actually does not belongs to the Grassmannian Gr(2,7). To see this recall that the Plücker equations for the Grassmannian Gr(2,7) are obtained by picking the 4-Pfaffians of the 7x7 skew-symmetric matrix

$$\begin{pmatrix} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} \\ & & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} \\ & & & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} \\ & & & & x_{4,5} & x_{4,6} & x_{4,7} \\ & & & & & x_{5,6} & x_{5,7} \\ & & & & & & x_{6,7} \\ & & & & & & \end{pmatrix}$$

where 4-Pfaffians means that we have to remove everytime 3 rows and column, indexed by the same triple. As an example, we might delete row and column $\{5,6,7\}$ and being left with

$$\begin{pmatrix} x_{1,2} & x_{1,3} & x_{1,4} \\ & x_{2,3} & x_{2,4} \\ & & x_{3,4} \end{pmatrix}$$

with the usual rule

$$\mathrm{Pf}_{567} = x_{1,2}x_{3,4} - x_{1,3}x_{2,4} - x_{1,4}x_{2,3}.$$

By looking at the action of τ_7 any point (of non-coordinate type) of the form 6.4 can have either two or three non-zero coordinates, with the sum of the indeces being constant

mod.7. Call these $(i,j),(k,h),(r,s)$. Substituting in the Plücker in both case we will have either a surviving (say) $x_{i,j}x_{r,s}=0$ or all three possibilities. In both cases, this implies that none of these points belongs to the Grassmannian.
What happens now with of σ_7? Recall the construction in 6.3. The fixed locus of the involution on the ambient $\mathbb{P}^{12}$ is given by the disjoint union of $\mathbb{P}^+ \sqcup \mathbb{P}^-$, with

$$\mathbb{P}^+ = V(x_{3,7}-x_{2,6},\dots,x_{2,4}-x_{5,7})$$

and

$$\mathbb{P}^- = V(x_{3,7}-x_{2,6},\dots,x_{2,4}-x_{5,7},x_{4,5}).$$

Intersecting with the 35 Pfaffians this gives us the union of $C_1 \sqcup C_2$ with C_1 being 10 points and C_2 a smooth plane conic. All the other six (conjugate) involutions yields the same type of fix locus. The result follows. □

The Y-model Computations here are identical to the Z model, and yields the same results. One has just to verify that τ_6 yields (on the fourfold Y_λ) the points $p_{1,6}$, $p_{2,5}, p_{3,4}$ whereas the fixed locus of the involutive part comes from the intersection with the zero set of the equations $\{x_{i,j} \pm x_{7-j,7-i}\}$, and the same for the other conjugate involutions. We just want to remark that even Y admits a concrete description in terms of equations in $\mathbb{P}^{14}$. Recall from 6.1 the description of Y_λ as

$$Y_\lambda = \{< a,b >\in \mathrm{Gr}(2,6) \mid \lambda(a,b) \in < a,b >\}.$$

If $a_1,\dots,a_6$ and $b_1,\dots,b_6$ denotes the coordinates of a,b with respect to the standard basis fixed, and if we call $p_{i,j}=a_ib_j-b_ia_j$ the Plücker coordinates in $\bigwedge^2 V_6$ the condition above translates in matrix form as

$$\mathrm{rk}\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ p_{2,6}+p_{3,5} & p_{3,6}+p_{4,5} & p_{1,2}+p_{4,6} & p_{1,3}+p_{5,6} & p_{1,4}+p_{2,3} & p_{1,5}+p_{2,4} \end{pmatrix} = 2.$$

Expanding the determinant in Laplace forms one gets quadratic equation in the dual of the Plücker coordinates, getting in this way Y_λ as explicit subvariety of $\mathbb{P}^{14}$.

6.1.3 Quotient Calabi-Yau threefold and surface of general type with an involution

The analysis in the previous paragraph shows how the fixed locus of the dihedral group D_7 depends only on the seven conjugate involutions. In particular the normal subgroups $\mathbb{Z}/7 \triangleleft D_7$ yields a free action on each member of the invariant family, both in the Calabi-Yau and in the surface case. We can then take the quotient for such subgroup and produce new families of varieties in dimension (respecively) 2 and 3. Since we can perform the construction in both Y and Z model, we will simply write W and S.

Theorem 6.1.8. *Let W a linear section of the Grassmannian Gr(2,7) constructed as above. Then W admits a free $\mathbb{Z}/7$ action. In particular the quotient $\pi : W \to \widetilde{W}$ yields a smooth Calabi-Yau threefold.*

Proof. Follows from description in lemma 6.1.7, where an explicit description of the fixed locus of the dihedral group on W is given. □

Corollary 6.1.9. *The Calabi-Yau $\widetilde{W}$ has Euler characteristic $\chi(\widetilde{W}) = -14$. In particular the Hodge diamond of $\widetilde{W}$ is*

$$\begin{array}{ccccccc} 1 & & 8 & & 8 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Corollary 6.1.10. *Let $\widetilde{S}$ the surface of general type obtained by intersecting $\widetilde{W}$ with a $\mathbb{Z}/7$-invariant hyperplane section. Then $p_g(\widetilde{S}) = 1, q(\widetilde{S}) = 0$, $K^2_{\widetilde{S}} = 6$. In particular its Hodge diamond is*

$$\begin{array}{ccccc} 1 & & 14 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

As one can see from 6.3 the surface construction depends by 8 parameters. Moreover, the whole family is unobstructed. Indeed already on the level of S one verifies

Lemma 6.1.11. *Let S_Z be a codimension 8 (linear) complete intersection in the Grassmannian Gr(2,7). Then $H^2(S_Z, T_{S_Z}) = 0$.*

Proof. To S_Z is associated the standard tangent sequence

$$0 \to T_{S_Z} \to T_{\mathrm{Gr}}|_{S_Z} \to (\mathcal{O}_{S_Z}(1))^{\oplus 8} \to 0.$$

Passing in cohomology we get

$$\ldots \to 0 \to (H^1(\mathcal{O}_{S_Z}(1))^{\oplus 8} \to H^2(S_Z, T_{S_Z}) \to H^2(S_Z, T_{\mathrm{Gr}}|_{S_Z}) \to \ldots$$

Since $(H^1(\mathcal{O}_{S_Z}(1))^{\oplus 8} = 0$, the claim will be proved if $H^2(S_Z, T_{\mathrm{Gr}}|_{S_Z}) = 0$. To prove this, first we realize that, thanks to the standard pairing

$$\Omega^k_{\mathrm{Gr}} \otimes \Omega^{N-k}_{\mathrm{Gr}} \to \omega_{\mathrm{Gr}}$$

we have $T_{\mathrm{Gr}} \cong \Omega^9(7)_{\mathrm{Gr}}$. We then use the Koszul complex for a complete intersection in a Grassmannian after tensoring with T_{Gr}. In particular we have

$$\cdots \to (T_G(-1))^8 \to T_{\mathrm{Gr}} \to T_{\mathrm{Gr}}|_{S_Z} \to 0.$$

Splitting in short exact sequences, we have that we will have vanishing of $H^2(S_Z, T_{\mathrm{Gr}}|_{S_Z})$ if both $H^2(T_{\mathrm{Gr}})$ and $H^3((T_{\mathrm{Gr}}(-1))$ does the same. But these are isomorphics to (resp.) $H^2(\mathrm{Gr}, \Omega^9(7))$ and $H^3(\mathrm{Gr}, \Omega^9(6))$, and these vanishing are authomatic for the Grassmannian Gr(2,7) (see [91], lemma 0.1). $\square$

Notice that the surface $\widetilde{S}$ comes with a involution $\sigma : \widetilde{S} \to \widetilde{S}$. Surfaces of general type with an involution are widely studied, see for example [33]. The fixed locus of the involution σ consists in one smooth plane conic C and 10 nodes. We can take the quotient $\sigma : \widetilde{S} \to \widetilde{S}/\sigma =: \Sigma$. By adjunction formula $K_{\widetilde{S}} = \sigma^*(K_\Sigma) + C$: therefore

$$K^2_\Sigma = \frac{K^2_{\widetilde{S}} + C^2 - 2K_{\widetilde{S}}C}{2}$$

One compute $K_{\widetilde{S}}C = 2$; moreover the adjunction formula for curves on a surface says $K_{\widetilde{S}}C + C^2 + 2\chi(\mathcal{O}_C) = 0$: this yields $C^2 = -4$ and

$$K^2_\Sigma = -1.$$

Σ is thus a surface with $k(S) = -\infty$. It would be interesting to study in more details the property of the pair $(\widetilde{S}, \sigma)$.

We point out that the original aim of the construction was to produce an example of a surface of general type with $p_g = q = 0, K^2 = 3$, and a fundamental group of order 14. The simplest way to produce it was to cook up a fix-point-free involution on our surface. As we have seen, the involution σ has indeed a fixed locus, making impossible to extend this construction any further.

Nevertheless, there might be some hope to associate to $\widetilde{S}$ a surface of general type with geometric genus 0. For example the idea could be to produce a suitably nice degeneration of $\widetilde{S}$ (for example, by imposing all $\mu_i = 1$), and try to mimic the construction of Barlow in [8], building a double cover branched on the set of nodes. However such a strategy necessarily pass through the explicit determination of the singular locus of $S_{\deg}$, and we have still to understand this properly.

6.1.4 Pfaffian-Calabi Yau correspondence and the Reid $\mathbb{Z}/7$-Campedelli surface

Our construction is closely related to another famous minimal surface of general type, the $\mathbb{Z}/7$ Campedelli-Reid surface from [101] (we will call them *quotient linked*). This goes via another well known geometric construction, the *Pfaffian-Calabi Yau correspondence*, considered by many authors in [103], [18].
Before making everything explicit, we recall the two main ingredients of the construction.

The Pfaffian-Grassmannian equivalence

We want to describe now another Calabi-Yau $W^{\vee}$ related to our W. We will follow closely the description of Borisov-Caldararu in [18]. Let fix V as the vector space of dimension 7. If $W \subset \mathrm{Gr}(2,7) \subset \mathbb{P}(\bigwedge^2 V) \cong \mathbb{P}^{20}$, take the dual projective space

$$\mathbb{P}^* = \mathbb{P}(\wedge^2 V^*)$$

as the projectivization of the space of two-forms on V. The Pfaffian locus

$$\mathrm{Pf} \subset \mathbb{P}^*$$

is defined to be the projectivization of the locus of degenerate two-forms on V (forms of rank ≤ 4). Equations for Pf can be obtained by taking the Pfaffians of the diagonal minors of a skew-symmetric 7×7 matrix of linear forms on V. Note that this yields cubic equation.

While the Grassmannian G is smooth, the Pfaffian Pf is a singular subvariety of $\mathbb{P}^*$ of dimension 17. Indeed, a point $\omega \in \mathrm{Pf}$ will be singular precisely when the rank of ω is two.

The Pfaffian is the classical projective dual of the Grassmannian:

$$\mathrm{Pf} = \{y \in \mathbb{P}^* \ : \ \mathrm{Gr} \cap H_y \text{ is singular}\},$$

where H_y is the linear space in $\mathbb{P}$ corresponding to y.

Consider a seven-dimensional linear subspace

$$H^\vee \subset \wedge^2 V^*,$$

and by abuse of notation $H^\vee$ will denote its image in $\mathbb{P}^*$ as well. Let $W^\vee$ be the intersection of $H^\vee$ with Pf.

On the dual side, let

$$H = \mathrm{Ann}(H^\vee) \subset \wedge^2 V$$

be the 14-dimensional annihilator of $H^\vee$; and W be the intersection of H and Gr. From the construction is evident that $W^\vee$ is the projective dual of the W we started from. Actually one has even more, namely

Theorem 6.1.12 (Thm 0.3 in [18])**.** *For a given choice of H, if either W or $W^\vee$ has dimension three, then W is smooth if and only if $W^\vee$ is. When this happens there exists an equivalence of derived categories*

$$\Phi : \mathrm{D}(W) \xrightarrow{\sim} \mathrm{D}(W^\vee).$$

Note that W and $W^\vee$ are not even birational, since they have Picard rank $\rho = 1$ and different degrees (respectively 42 and 14).
Actually, even if worth mentioning, we are not going to use the derived part of the picture. We will indeed just use the classical projective duality between W and $W^\vee$.

The Campedelli-Reid $\mathbb{Z}/7$ surface

Recall the construction of the $\mathbb{Z}/7$ Campedelli-Reid surface from [101].
The aim is constructing a canonically embedded and projectively Cohen-Macaulay surface of general type $V \subset \mathbb{P}^5$ with $p_g = 6,\ K^2 = 14$. These hypotheses implies that the coordinate ring

$$R(V, K_V) = \bigoplus_{m \geq 0} H^0(V, mK_V)$$

is Gorenstein and of codimension 3. In particular, by the famous structure theorem of Buchsbaum-Eisenbud, the ideal of relation can be written as submaximal Pfaffians of a 7×7 skew matrix.
This means that there is a skew 7×7 matrix $M = (l_{ij})$ with entries l_{ij} linear forms in

the homogeneous coordinates of $\mathbb{P}^5$, and the ideal I_Y is generated by the 7 cubic forms Pf_i obtained as the 6×6. That is, delete the ith row and jth column of M to obtain a skew 6×6 skew Matrix M_{ij} ; then the determinant of M_{ij} is a product of two Pfaffians:

$$\det M_{ij} = \pm \mathrm{Pf}_i \, \mathrm{Pf}_j$$

as a polynomial identity in the entries of M, and in particular, the diagonal minors are perfect squares: $\det M_{ii} = \mathrm{Pf}_i^2$. One shows that if the entries l_{ij} of M are sufficiently general then $V : (\mathrm{Pf}_i = 0)$ has the stated properties.

Our purpose now is to construct a free action of the group $\mathbb{Z}/7$ on V. The general V will not be $\mathbb{Z}/7$-invariant, but we can still get an invariant subfamily by choosing M carefully. To do this, let us define an action of $\mathbb{Z}/7$ on $\mathbb{P}^5$ by $x_i \mapsto \varepsilon^i x_i$. We will have

$$M = \begin{pmatrix} 0 & x_1 & x_3 & x_2 & x_6 & x_4 & x_5 \\ & 0 & x_4 & \lambda_3 x_3 & 0 & -\lambda_5 x_5 & -x_6 \\ & & 0 & x_5 & \lambda_2 x_2 & 0 & -\lambda_1 x_1 \\ & & & 0 & x_1 & \lambda_6 x_6 & 0 \\ & -\mathrm{sym} & & & 0 & x_3 & \lambda_4 x_4 \\ & & & & & 0 & x_2 \\ & & & & & & 0 \end{pmatrix}$$

Any of the 6 Pfaffians will be of the type

$$\mathrm{Pf}_i = \sum_{\substack{j+k+l \equiv \\ i \ mod \ 7}} \alpha_{jkl} x_j x_k x_l;$$

so for example

$$\mathrm{Pf}_1 = -\lambda_5 x_5^3 - \lambda_1 \lambda_6 x_1^2 x_6 + (1 - \lambda_1 \lambda_5) x_1 x_2 x_5 - \lambda_1 \lambda_3 x_1 x_3 x_4 + x_2 x_4^2 +$$

$$-\lambda_3 x_2 x_3^2 + (1 + \lambda_6) x_4 x_5 x_6 + \lambda_6 x_3 x_6^2 \tag{6.5}$$

From the construction $\mathrm{Pf}_i \mapsto \varepsilon^i \, \mathrm{Pf}_i$. Moreover for sufficiently general values of λ_i the surface $V = V(\mathrm{Pf}_0 = \ldots = \mathrm{Pf}_6 = 0)$ is smooth, and therefore one has

Theorem 6.1.13 ([101])**.** *Pick M as above, and $V \subset \mathbb{P}^5$ the corresponding surface. The quotient $\widetilde{V} = V/\mathbb{Z}/7$ is a smooth surface of general type with $p_g = q = 0$, $K^2 = 2$, that is a Campedelli surface.*

One has the following

Question 6.1.14. *Does there exist a smooth Calabi-Yau threefold $\widetilde{U}$ extending $\widetilde{V}$?*

From our surface to the Campedelli-Reid

Consider now the quotient Calabi-Yau $\widetilde{W}$ constructed in 6.1.8, and let now $\widetilde{W}^\vee$ the dual variety to $\widetilde{W}$ as constructed above. Denote by $\mathrm{Pf}(V)$ the Pfaffian variety in $\mathbb{P}^{20}$. We have the following answer to question 6.1.14

Proposition 6.1.15. *$\widetilde{W}^\vee$ is the extension to a Calabi-Yau threefold of the Campedelli-Reid $\mathbb{Z}/7$ surface. In particular if H_7 is a $\mathbb{Z}/7$-invariant hyperplane section one has $\widetilde{W}^\vee \cap H_7 = \widetilde{V}$, with $\widetilde{V}$ as in the section above.*

Proof. Recall that the equations of a $\mathbb{Z}/7$-invariant W are the one listed in 6.2, to which we have to add one further linear equation in the variables $x_{2,7}, x_{3,6}, x_{4,5}$ (corresponding to the 0-eigenspace). In particular such seven equation will form a seven-dimenisonal linear subspace $\mathbb{P}(\Lambda) \subset \mathbb{P}(\bigwedge^2 V^*)$. Equations for the dual variety $W^\vee$ can be then obtained by considering $\mathbb{P}(\Lambda^\perp)$. Note that this gives us 14-codimensional linear section of the Pfaffian variety, grouped by their eigenvalue with respect of the $\mathbb{Z}/7$ action. For example we will have

$$W^\vee = V(x_{1,2} - \mu_1 x_{3,7}, x_{3,7} - \mu_2 x_{4,6}, \dots) \subset \mathrm{Pf}(V),$$

and so on according to the same rule. Therefore we can project down to the $\mathbb{P}^6$ with coordinates $x_{1,2}, \dots, x_{1,6}, x_{2,7}$, where we chose one representative for any eigenspace. The dual variety obtained $W^\vee$ will be smooth if only if W is so by [18]. Anyway, since the codimension is small, we can directly check the smoothness of $W^\vee$ by any computer algebra system. One can see directly that the equation for $W^\vee$ can arranged in Pfaffian format inside the matrix

$$M = \begin{pmatrix} 0 & x_{1,2} & x_{1,4} & x_{1,3} & x_{1,7} & x_{1,5} & x_{1,6} \\ & 0 & x_{1,5} & \lambda_3 x_{1,4} & x_{2,7} & -\lambda_5 x_{1,6} & -x_{1,7} \\ & & 0 & x_{1,7} & \lambda_2 x_{1,4} & x_{2,7} & -\lambda_1 x_{1,3} \\ & & & 0 & x_{1,3} & \lambda_6 x_{1,7} & x_{2,7} \\ & -\mathrm{sym} & & & 0 & x_{1,5} & \lambda_4 x_{1,5} \\ & & & & & 0 & x_{1,3} \\ & & & & & & 0 \end{pmatrix}$$

with appropriate parmeters. By the same argument of [101] one has that the Pfaffians are $\mathbb{Z}/7$ invariant, and therefore realize the quotient $\widetilde{W}^\vee$. Moreover notice that by picking one further $x_{2,7} = 0$ one gets down exactly to the equations of the $\mathbb{Z}/7$ Campedelli-Reid surface described in 6.1.4. $\square$

6.1.5 Further invariance property: Frobenius group of order 21

The dihedral group D_7 is not the biggest group under which the family of surfaces is invariant. To see this, let us rewrite $S \subset \mathbb{P}^{12}$ in a way inspired by Reid's construction of the $\mathbb{Z}/7$ Campedelli surface described in the previous section. Namely, pick coordinates $x_1, \ldots, x_6, y_1, \ldots, y_6, z$ and define

$$S = V(\mathrm{Pf}(4, M))$$

with

$$M = \begin{pmatrix} 0 & x_1 + y_1 & x_3 + y_3 & x_2 + y_2 & x_6 + y_6 & x_4 + y_4 & x_5 + y_5 \\ & 0 & x_4 & \lambda_3 y_3 & z & -\lambda_5 y_5 & -x_6 \\ & & 0 & x_5 & \lambda_2 y_2 & z & -\lambda_1 y_1 \\ & & & 0 & x_1 & \lambda_6 y_6 & z \\ & -\mathrm{sym} & & & 0 & x_3 & \lambda_4 y_4 \\ & & & & & 0 & x_2 \\ & & & & & & 0 \end{pmatrix}$$

Denote by a the cyclic generator sending $x_i \mapsto \varepsilon^i x_i$, $y_i \mapsto \varepsilon^i y_i$, $z \mapsto z$ and b the generator sending $x_i \mapsto x_{2i}$, $y_i \mapsto y_{2i}$, $z \mapsto z$. This corresponds to the cycle $(2,4,6)(3,5,7)$. Denote by F_{21} the group (of order 21) generated by a, b. One checks that $ab = b^2 a$. Therefore by the classification of small groups, F_{21} is isomorphic to the Frobenius group of order 21, which can be represented as the subgroup of S_7 generated by $(2,3,5)(4,7,6)$ and $(1,2,3,4,5,6,7)$, and is the Galois group of $x^7 - 14x^5 + 56x^3 - 56x + 22$ over the rationals. The fixed locus is given by imposing $x_1 = \rho^i x_2 = \rho^{2i} x_4$ (and so on for the other coordinates), where ρ is a third root of unity. It consists of 6 points.

We point out that the family is invariant under the group G_{42} of order 42 generated by a and b', with $b' : x_i \mapsto x_{2i}$. This construction can be adapted in a straighforward way from the one already given in [101].

6.1.6 Another D_7 action

The dihedral action we defined is not the only one that can be constructed on the Grassmannian. Indeed we may specify a point in the Grassmannian $\mathrm{Gr}(k, n)$ as a $k \times n$ matrix. The symmetric group S_n then acts permutating the columns. Thus the dihedral subgroup D_n of S_n generated by the n-cycle $\alpha : (1, 2, \ldots, n)$ and the longest element in the group w_0. The latter, in the case of the symmetric group, corresponds to the permutation $i \mapsto n + 1 - i$.

In this case the involutions corresponds to our original one, while the order seven element comes from the discussion in the above subsection. In more concrete terms, define $S \subset \mathbb{P}^{12}$ be the zero set of the linear equation

$$H = \sum_{i=1}^{7} \lambda x_i + \sum_{i=1}^{7} \mu y_i$$

and the 4-Pfaffians of the matrix

$$M = \begin{pmatrix} 0 & \lambda x_6 + \mu y_6 & \lambda x_2 & \mu x_5 & \mu y_1 & \lambda x_4 & \lambda x_7 + \mu y_7 \\ & 0 & \lambda x_5 + \mu y_5 & \lambda x_1 & \mu y_4 & \mu y_7 & \lambda x_3 \\ & & 0 & \lambda x_4 + \mu y_4 & \lambda x_7 & \mu y_3 & \mu y_6 \\ & & & 0 & \lambda x_3 + \mu y_3 & \lambda x_6 & \mu y_2 \\ & -\mathrm{sym} & & & 0 & \lambda x_2 + \mu y_2 & \lambda x_5 \\ & & & & & 0 & \lambda x_1 + \mu y_1 \\ & & & & & & 0 \end{pmatrix}$$

The action of the 7-cycle α sends $x_1 \mapsto x_2 \mapsto \ldots \mapsto 7 \mapsto 1$ for both x_i and y_i, while w_0 sends $x_1 \mapsto x_6$, $x_2 \mapsto x_5$ and $x_3 \mapsto x_4$, keeping x_7 fixed (and similar for y_i). The surface defined above is clearly invariant under this new dihedral action: however, if we compute the fix locus we got the same answer of the old model (that is, a smooth conic and 10 isolated points).

6.2 A similar phenomenon in Gr(3,6)

6.2.1 A digression from character theory

As hinted in the previous section, the main motivation for this construction was to find an example of a surface of general type with $p_g = q = 0$, $K^2 = 3$. The action defined in the previous section failed to have a fix-point-free action on the involutive part. Nevertheless, some hint from representation theory motivated us to look further. Denote as above W and S for the Calabi-Yau threefold and the surface of degree 42, and let A be the ample divisor coming from the Plücker embedding. One has in particular that on S, $A = K_S$. Denote by V_{reg} the regular representation of the dihedral group D_7.

One has

$$H^0(W, A) = V_{\text{reg}}$$

and

$$H^0(S, K_S) = V_{\text{reg}} \ominus \mathbb{C},$$

where $\mathbb{C}$ here denotes the trivial representation. If we consider $H^0(W, 2A)$ this has to be a multiple of V_{reg} (and the same for S). In particular the map

$$\varphi : \operatorname{Sym}^2(H^0(S, K_S)) \longrightarrow H^0(S, 2K_S) = 4V_{\text{reg}}$$

has to be surjective, the kernel denoting the quadrics through S . Denote by $\mathbb{1}$ the trivial representation of D_7, ι the (1-dimensional) sign representation and μ_1, μ_2, μ_3 the three two-dimensional irreducible representation. Recall that in term of characters one has

$$\chi_{\operatorname{Sym}^2 V_{\text{reg}}(g)} = \frac{1}{2}(\chi(g)^2 - \chi(g^2)).$$

Computing characters one has

$$4V_{\text{reg}} = \mathbb{1}^{\oplus 4} \oplus \iota^{\oplus 4} \oplus \mu_1^{\oplus 8} \oplus \mu_2^{\oplus 8} \oplus \mu_3^{\oplus 8}$$

and

$$\operatorname{Sym}^2(V_{\text{reg}} \ominus \mathbb{C}) = \mathbb{1}^{\oplus 7} \oplus \iota^{\oplus 6} \oplus \mu_1^{\oplus 13} \oplus \mu_2^{\oplus 13} \oplus \mu_3^{\oplus 13}.$$

The kernel φ is thus generated by $\mu_1^{\oplus 5}, \mu_2^{\oplus 5}, \mu_3^{\oplus 5}, \iota^{\oplus 2}, \mathbb{1}^{\oplus 3}$. This suggests that out of the 35 quadrics through S, 30 should be exchanged two by two by the two-dimensional irreducible subrepresentations, 3 preserved and 2 exchanged by a sign. The equations of Gr(2,7) does not satisfies this pattern. As we are going to see in the next section, there is another candidate anyway who seems more suitable.

6.2.2 Invariant surface family in the Grassmannian Gr(3,6)

The Grassmannian $\operatorname{Gr}(3, 6)$ shares many numerical similiarities with the Grassmannian $\operatorname{Gr}(2, 7)$. First of all notice how the Plücker spaces have very similar dimensions

$$\varphi : \operatorname{Gr}(3, 6) \longrightarrow \mathbb{P}(\bigwedge^3 V_6) \cong \mathbb{P}^{19}.$$

Moreover the dimension of the Grassmannian $\operatorname{Gr}(3, 6)$ is 9, and defined exactly by 35 Plücker quadrics. Both Grassmannians have degree equals to 42. Of course $\operatorname{Gr}(3, 6)$ is not an hyperplane section of $\mathrm{Gr}(2,7)$, nevertheless a further (and even more relevant) similarity comes from their Hilbert-Poincaré Series. One has in fact

$$\mathrm{HP}(\mathrm{Gr}(3,6) = \frac{P(t)}{(1-t)^{19}};\ \mathrm{HP}(\mathrm{Gr}(2,7) = \frac{P(t)}{(1-t)^{20}},$$

with the same Hilbert numerator $P(t)$.

Consider now a eight-codimensional linear section of the Grassmannian $\mathrm{Gr}(2,7)$ and a seven-codimensional linear sections of the Grassmannian $\mathrm{Gr}(3,6)$. The first one is the already considered S_{42}, and let us call T the second one. Of course both S and T by Lefschetz theorem are regular surface, of degree 42 and by adjunction their canonical class $\omega \cong \mathcal{O}(1)$. Moreover, since the Hilbert numerators are the same for both Grassmannians, they have the same numerical invariants. The idea is try to replicate the D_7 construction on the $\mathrm{Gr}(3,6)$ model. Note that the same construction cannot extend to the Calabi-Yau case in dimension 3. Indeed, as in [71] a 6-codimensional linear section in $\mathrm{Gr}(3,6)$ has Euler characteristic -96, ruling out even the possibility of any fix-point-free action of a group with order divisible by seven.

As before, we have to build up a D_7 action on V_6 and later on extend to the Grassmannian. Let us define this action by sending

$$x_i \mapsto \varepsilon^i x_i, \quad x_i \mapsto x_{6-i}.$$

This action extends to $\bigwedge^3 V_6$ in the obvious way, with

$$x_{i,j,k} \mapsto \varepsilon^{i+j+k} x_{i,j,k}, \quad x_{i,j,k} \mapsto -x_{6-i,6-j,6-k}.$$

It is easy to see that the Grassmannian Gr(3,6) is preserved under this action. The problem reduces then to find an invariant $\mathbb{P}^{12}$, as in the previous cases. Observe now that any $\mathbb{Z}/7$ eigenvalue different from zero can be obtained in three distinct way as sum mod 7 of strictly increasing natural numbers between 1 and 6. For example $1 \equiv 1+2+5 \equiv 1+3+4 \equiv 4+5+6$ and so on. Zero behaves differently, since we have only $0 \equiv 1+2+4 \equiv 3+5+6$. We can therefore build up equations for T by picking

$$T = V(\ldots, \sum_{i+j+k \equiv c} \alpha_{i,j,k} x_{i,j,k}, \ldots).$$

Choosing the $\alpha_{i,j,k} = \alpha_{6-i,6-j,6-k}$ we immediately obtain not only the $\mathbb{Z}/7$ invariance but the full D_7 as well. The quadratic equations in this invariant $\mathbb{P}^{12}$ seems to behave

better with respect to the calculations in the above subsection: indeed we have thirty made by three terms, and five split in four and six terms.
By doing computations totally similar to the one in the Gr(2,7) case one shows that the $\mathbb{Z}/7$ part of the action is free. Each conjugate involution fixes an elliptic curve E of degree 6 and 6 distinct points. In particular by adjunction formula

$$K_T \cdot E = 6 \Rightarrow E^2 = -6$$

and

$$K^2_{T/\sigma} = \frac{6 + E^2 - 2K_T \cdot E}{2} = -6.$$

We point out that we have not been able to check the smoothness of T for generic coefficients without appealing to a *tour-de-force* in computational algebra. We can state anyway the following proposition.

Proposition 6.2.1. *Let T a smooth surface constructed as above. The quotient $T/\mathbb{Z}/7$ is a smooth surface of general type with $p_g = 1, q = 0, K^2 = 6$, together with an involution σ.*

6.3 Further invariant familes and future directions

It seems that none of the model analysed here can give rise to fix-point-free action of the dihedral group D_7. Nevertheless, the hunt for a surface of general type with $p_g = q = 0, K^2 = 3$ is not over. Indeed, there could be some more involutions that we have not detected yet.
We suspect that a successfull approach might come from representation theory, as in [42] . Indeed, the 4- Pfaffians of a 7x7 skew matrix of linear forms are indeed equations for the affine Grassmannian $a\,\mathrm{Gr}(2,7)$. Under the induced action of $\mathrm{GL}(V)$ on $\bigwedge^2 V$, the scalar matrices $\lambda \cdot I$ act by λ^2. However, the straight Grassmannian $\mathrm{Gr}(2,7) \subset \mathbb{P}^{20}$ is the quotient of $a\,\mathrm{Gr}(2,7)$ by $\mathbb{C}^*$ acting on $\bigwedge^2 V$ by overall scalar multiplication by $\mu \in \mathbb{C}^*$ and this is not covered by the $\mathrm{GL}(V)$ action; the full symmetry group is thus a double cover of $\mathrm{GL}(V)$ (an index 2 central extension). This might indeed explain the sign problem we experienced and eventually solve our problem.
Another approach might be linked to $\mathbb{Z}/14$, maybe in connection with the index 2 symmetry explained above. The easiest possible action to define on V_7 is $\frac{1}{14}(1,2,3,4,5,6,7)$. This in turn becomes an action on $\bigwedge^2 V$, with the Grassmannian invariant. However,

the ring of invariants

$$\oplus_k [H^0(\mathrm{Gr}(2,7), \mathcal{O}_G(k))]^{\mathbb{Z}/14}$$

is not as nice as in the $\mathbb{Z}/7$ case. Indeed if we regroup the monomial basis of $H^0(\mathrm{Gr}(2,7), \mathcal{O}_G(1))$ according to the eigenvalues for the $\mathbb{Z}/14$ action we get the following

3	4	5	6	7	8	9	10	11	12	13
$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	$x_{1,6}$	$x_{1,7}$	$x_{2,7}$	$x_{3,7}$	$x_{4,7}$	$x_{5,7}$	$x_{6,7}$
		$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$	$x_{3,6}$	$x_{4,6}$	$x_{5,6}$		
				$x_{3,4}$	$x_{3,5}$	$x_{4,5}$				

We have to choose an invariant $\mathbb{P}^{12}$, that is eight equations invariant under the group, without any natural choice as in the $\mathbb{Z}/7$ case.
If we consider in a dual way the fix points for the $\mathbb{Z}/14$ action, this consists of three $\mathbb{P}^2$ (with variables - respectively- $\{x_{1,6}, x_{2,5}, x_{3,4}\}$, $\{x_{1,7}, x_{2,6}, x_{3,5}\}$, $\{x_{2,7}, x_{3,6}, x_{4,5}\}$), four $\mathbb{P}^1$ (with variables - respectively - $\{x_{1,4}, x_{2,3}\}$, $\{x_{1,5}, x_{2,4}\}$, $\{x_{3,7}, x_{4,6}\}$, $\{x_{4,7}, x_{5,6}\}$) and four distinct points (the coordinate points $x_{1,2}, x_{1,3}, x_{5,7}, x_{6,7}$).
The most natural choices for the eight equation are a linear combination of the three coordinates with eigenvalue $7, 8, 9$, four with eigenvalue $5, 6, 10, 11$ and one of the remaining others. In this way three points in the resulting S are fixed. For other choices of equations we get either 6 or 9 fixed points. The problem is that any S chosen this way is singular on a quintic curve. On the other hand if we choose the eight equation as (separate) linear combination of coordinate with eigenvalues $7, 8, 9$ and $5, 6, 10, 11$ the singular locus has dimension 0, and there is therefore a hope for the construction to work.

6.3.1 List of candidates

From the list of [71] many other families of surfaces and Calabi-Yau threefolds seems to have the right numerology to admit action of finite groups. We list here the most promising examples we want to work on next. We write W for the threefold, S for the surface, G for the most promising group we identified. The sheaf is the one used to construct the threefold W: to get the surface, one needs to cut down with a further copy of $\mathcal{O}(1)$.

$G(k,n)$	$\mathcal{F}$	$K^{\dim}$	$\chi(W)$	$\chi(S)$	G	Model
$\mathrm{Gr}(2,5)$	$\mathcal{O}(1) \oplus 2\mathcal{O}(2)$	20	-120	88	$\mathbb{Z}/4$	
$\mathrm{Gr}(2,5)$	$2\mathcal{O}(1) \oplus O(3)$	15	-130	81	$\mathbb{Z}/3$	
$\mathrm{Gr}(2,5)$	$\bigwedge^2 \mathcal{Q}(1)$	25	-100	95	$\mathbb{Z}/5$	
$\mathrm{Gr}(2,6)$	$\bigwedge^3 \mathcal{Q} \oplus \mathcal{O}(3)$	18	-162	90	$\mathbb{Z}/6, S_3$	$(\mathbb{P}^2 \times \mathbb{P}^2)_3$
$\mathrm{Gr}(2,6)$	$\mathcal{S}^*(1) \oplus \mathcal{O}(1)^{\oplus 3}$	33	-102	11	$\mathbb{Z}/3$	
$\mathrm{Gr}(2,6)$	$\mathrm{Sym}^2 \mathcal{S}^* \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$	40	-32	32	D_4	$(\mathbb{P}^3 \times \mathbb{P}^3)_{1^2,2}$
$\mathrm{Gr}(2,6)$	$\mathrm{Sym}^2 \mathcal{S}^* \oplus \mathcal{S}^*(1)$	48	-92	132	$\mathbb{Z}/4$	
$\mathrm{Gr}(2,7)$	$\mathrm{Sym}^2 \mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 4}$	56	-92	148	$\mathbb{Z}/2$	
$\mathrm{Gr}(2,7)$	$(\mathrm{Sym}^2 \mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 4}$	80	-16	148	$\mathbb{Z}/4$	
$\mathrm{Gr}(2,7)$	$(\mathrm{Sym}^2 \mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 4}$	80	-16	148	$\mathbb{Z}/4$	
$\mathrm{Gr}(2,7)$	$(\bigwedge^4 \mathcal{Q}) \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$	36	-120	120	$\mathbb{Z}/6, S_3$	$(G_2/P_1)_{1,2}$
$\mathrm{Gr}(2,7)$	$(\bigwedge^4 \mathcal{Q}) \oplus \mathcal{S}^*(1)$	42	-98	126	$\mathbb{Z}/7, \mathbb{Z}/14, D_7$	
$\mathrm{Gr}(2,8)$	$(\bigwedge^5 \mathcal{Q}) \oplus \mathcal{O}(1)^{\oplus 3}$	57	-84	147	$\mathbb{Z}/3$	
$\mathrm{Gr}(2,8)$	$(\bigwedge^5 \mathcal{Q}) \oplus \mathrm{Sym}^2 \mathcal{S}^*$	72	-72	168	$\mathbb{Z}/3, \mathbb{Z}/8$	
$\mathrm{Gr}(3,6)$	$(\bigwedge^2 \mathcal{S}^*) \oplus \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$	32	-116	56	$\mathbb{Z}/4$	
$\mathrm{Gr}(3,6)$	$(\bigwedge^2 \mathcal{S}^*) \oplus \mathcal{S}^*(1)$	42	-96	126	$\mathbb{Z}/7$	
$\mathrm{Gr}(3,7)$	$\mathrm{Sym}^2 \mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 3}$	128	-128	256	$\lvert G\rvert = 2^i$	$(\mathbb{P}^7)_{2^4}$
$\mathrm{Gr}(3,7)$	$\bigwedge^2 \mathcal{S}^* \oplus \bigwedge^3 \mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$	66	-84	168	$\mathbb{Z}/8, D_4$	
$\mathrm{Gr}(3,8)$	$(\bigwedge^2 \mathcal{S}^*)^{\oplus 2} \oplus \mathrm{Sym}^2 \mathcal{S}^*$	176	-64	304	$\mathbb{Z}/8, D_4, \mathbb{Z}/16, D_8$	
$\mathrm{Gr}(3,8)$	$(\bigwedge^2 \mathcal{S}^*)^{\oplus 4}$	92	-64	196	$\mathbb{Z}/4$	
$\mathrm{Gr}(3,8)$	$(\bigwedge^3 \mathcal{Q}) \oplus \mathcal{O}(1)^{\oplus 2}$	102	-84	210	$\mathbb{Z}/6$	
$\mathrm{Gr}(4,8)$	$\mathrm{Sym}^2 \mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 3}$	256	-256	256	$\lvert G\rvert = 2^i$	$(\mathbb{P}^7)_{2^4} \sqcup (\mathbb{P}^7)_{2^4}$
$\mathrm{Gr}(4,8)$	$(\bigwedge^2 \mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(2)$	48	-128	144	$\lvert G\rvert = 16$	$(\prod^4 \mathbb{P}^1)_2$
$\mathrm{Gr}(5,10)$	$(\bigwedge^2 \mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}$	120	-220	300	$\lvert G\rvert = 10$	$(\prod^5 \mathbb{P}^1)_{1^2}$

Table 6.1: Possible quotient surfaces and Calabi-Yaus

Chapter 7

Griffiths residues for hypersurfaces in Grassmannians

In this chapter, we define a Jacobian-like ring R^G_f for an hypersurface X in the Grassmannian $\mathrm{Gr}(k,n)$, and recover an equivalent of the Griffiths residue theory for R^G_f.

7.1 Generalised Jacobian ring and T^1

We define a notion of Jacobian ring for a smooth hypersurface in a Grassmannian. Our analysis comes from unraveling Mark Green's pseudo-Jacobian system (see [63], and [105] as well). It is important to notice that all results here hold as well for compact irreducible Hermitian symmetric spaces G/P. We will restrict nevertheless to the Grassmannian case, since the aim of this section is to get concrete results and practical algorithms for computing explicit Hodge groups.

Let us introduce some notation. If V_n is a $\mathbb{C}$-vector space of dimension n, we denote by $\mathrm{Gr}(k,V_n) = \mathrm{Gr}(k,n)$ the Grassmannian of k-planes in V_n. If there is no danger of confusion we will often write $\mathcal{O}_G$ for $\mathcal{O}_{\mathrm{Gr}(k,n)}$ (and similar for other sheaves). Recall that the Grassmannian $\mathrm{Gr}(k,n)$ is a smooth projective variety of dimension $N := k(n-k)$ closed in the Plücker embedding in $\mathbb{P}(\bigwedge^k V_n)$. Denote by $\mathcal{O}_G(1)$ the ample generator of $\mathrm{Pic}(\mathrm{Gr}(k,n)) \cong \mathbb{Z}$: we have in particular that $\omega_G \cong \mathcal{O}_G(-n)$. Denote by

$$S = \bigoplus_{a \geq 0} S_a, \quad S_a = H^0(\mathrm{Gr}(k,n), \mathcal{O}_G(a))$$

the homogeneous coordinate ring of the Grassmannian, and consider a hypersurface X of degree d, given by the vanishing of a generic section $f \in S_d$. We have the following definition

Definition 7.1.1 (cf. [63], [105])**.** The Jacobian ideal J_f of a smooth hypersurface $X = V(f)$ in $\mathrm{Gr}(k,n)$ is a homogeneous ideal of S generated by $f \in S_d$ and

$$\{v \cdot f \mid v \in \mathfrak{sl}_n \cong H^0(G, T_G)\}.$$

Denote by $R_f^G = S/J_f$ the corresponding *Jacobian ring*,

We want now to get a definition of R_f^G in a totally explicit way, that is in terms of generators and relations. To this, let us fix a basis $v_1, \dots, v_n$ for V_n and a dual basis $x_1, \dots, x_n$ for $V_n^\vee \cong \mathbb{C}[x_1, \dots, x_n]_1$. It is well known that

$$H^0(G, \mathcal{O}_G(1)) \cong \bigwedge^k V^\vee \cong \langle \cdots, x_I, \cdots \rangle,$$

where I denotes a multi-index $\{i_1, \cdots, i_k\}$ of $\{1, \cdots, n\}$ of length k, with $i_1 < \cdots < i_k$. In particular S is isomorphic to the Plücker algebra

$$S \cong \mathbb{C}[x_I]/P,$$

where x_I as above, and P denotes the ideal generates by the quadratic equations of the Plücker embedding. These can be computed quite easily in a recursive way, for example using Macaulay2.
To have a complete understanding of R_f^G we only have to make the $\mathfrak{sl}_n$ action explicit. There is a canonical action of $\mathfrak{sl}_n$ on the dual of its tautological module $(V_n)^\vee$ (cf. [93]). Recall that $\mathfrak{sl}_n$ is generated by

$$\{\ E_{i,j},\ E_{i,i} - E_{j,j} \mid i, j = 1, \dots, n, i \neq j\}$$

where $E_{i,j}$ denotes the matrix with one in the (i,j)-place and zeroes elsewhere. $E_{i,j}$ acts on $(V_n)^\vee$ as differential operator: more precisely to $E_{i,j}$ corresponds the derivations D_j^i defined by

$$D_j^i = x_i \frac{\partial}{\partial x_j}.$$

The action of D_j^i induces an action on $\bigwedge^k(V^\vee)$, by

$$D_j^i(x_{i_1} \wedge \cdots \wedge x_{i_k}) = D_j^i(x_{i_1}) \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} + \ldots + x_1 \wedge \cdots \wedge D_j^i(x_{i_k}).$$

For any r, one has

$$S_r < \mathrm{Sym}^r(H^0(G, \mathcal{O}_G(1))) \cong \mathrm{Sym}^r \bigwedge^k V^\vee.$$

The action of D^i_j can be extended to $\mathrm{Sym}^r \bigwedge^k V^\vee$ simply by Leibnitz's rule.
Therefore, if $X \subset \mathrm{Gr}(k,n)$ is given by the vanishing of a polynomial $f \in S_d$, J will be generated by f itself and by the n^2-1 degree d polynomial given by

$$\{\ D^i_j(f),\ D^i_i(f) - D^j_j(f) \mid i,j = 1,\dots,n, i \neq j\}. \tag{7.1}$$

We can then rephrase the definition of the Griffiths ring as follows.

Definition 7.1.2. Let $X = V(f)$ a smooth hypersurface in the Grassmannian Gr(k,n). Let S the coordinate ring of the (affine cone over the) Grassmannian, and let J the ideal of S generated by f and the equations in 7.1. We define the generalised Jacobian ring (or *Griffiths ring*) of X as

$$R^G_f := S/J.$$

Generalising Griffiths calculus, when appropriate vanishings are provided, the Hodge groups $H^p_{\mathrm{prim}}(\Omega^{n-p})$ are indeed contained in (some specific homogeneous component of) S. In particular there is a surjective map of graded rings

$$\bigoplus S_a \longrightarrow \bigoplus H^{p,n-p}_{\mathrm{prim}}(X).$$

Our purpose is to identify the kernel of this surjective map with the above defined Jacobian ideal J_f. Moreover, in what we consider being the core result of this section, we show how to give an explicit presentation of the Jacobian ring (and its graded components) in terms of generators and relations. This in turn allow us to recover explicit (polynomial) basis for the Hodge groups $H^{p,q}_{\mathrm{prim}}(X)$, in a generalisation of Griffiths's theorem on projective space.
We point out that the required vanishings to run Griffiths calculus programme not always work in the Grassmannian case. Nevertheless, we will give a generalised version of the Griffiths residue, showing how to effectively use our result in few distinguished examples.
The first step is to link the generalised Jacobian ring to the $T^1_{A_X}$ of the affine cone over X. Recall from chapter 3 that for a smooth projective hypersurface the module $T^1_{A_X}$ has actually a ring structure, and it is isomorphic, up to a shift, to the classical Jacobian ring of X. We want to show that the same happens for hypersurfaces in Grassmannian,

with the appropriate definition of the Jacobian ring given above.

Lemma 7.1.3. *Let X a smooth hypersurface of degree d in the Grassmannian $G = \mathrm{Gr}(k,n)$ defined by the vanishing of a $f \in H^0(G, \mathcal{O}_G(d))$. Let $dim(X) \geq 3$. Then we have an isomorphism*

$$T^1_{A_{X_d}}(-d) \cong R^G_f.$$

Proof. Consider the short exact sequence

$$0 \to T_X \to T_G|_X \to \mathcal{O}_X(d) \to 0.$$

For any twist with $\mathcal{O}_X(k)$ we consider the associated long exact sequence in cohomology on X

$$H^0(TG|_X(h)) \xrightarrow{\beta} H^0(\mathcal{O}_X(d+h)) \xrightarrow{\alpha} H^1(T_X(h)) \to H^1(TG|_X(h)). \tag{7.2}$$

The first thing to show is the vanishing of the last term in the sequence above. Indeed one uses the two standard exact sequences (for any k, t)

$$0 \to \Omega^k_G(t) \to \Omega^k_G(t+d) \to \Omega^k_G|_X(t+d) \to 0, \tag{7.3}$$

$$0 \to \Omega^{k-1}_X(t) \to \Omega^k_G|_X(t+d) \to \Omega^k_X(t+d) \to 0 \tag{7.4}$$

and the fact that $H^1(X, TG|_X) \cong (H^{N-1}(\Omega^1_G|_X(n-h))^\vee$. Indeed the latter is zero after expanding in cohomology the second sequence since by Proposition 2.4.1 in[105] we have the vanishing of $H^q(\Omega^1_G(t))$ for any $(q,t) \neq (1,0)$. Therefore in 7.2 by properties of exact sequences one has

$$H^1(X, T_X(h)) \cong H^0(X, \mathcal{O}_X(h+d))/\mathrm{Im}(\beta).$$

On the other hand the action of $H^0(TG|_X) \cong \mathfrak{sl}_n$ is given as the derivation action of $\mathfrak{sl}_n$ on the space of homogeneous polyonomial of degree h in the coordinate ring. This therefore coincides with the given definition of the Jacobian ring R^G_f. As in section 2, by $H^2(\mathcal{O}_X(h)) = 0$ one has

$$T^1_{A_X}(-d+h) \cong H^1(T_X(h)).$$

□

The above lemma implies in particular

$$(R_f^G)_0 \cong H^1(T_X), \quad (R_f^G)_m \cong H^{n-1,1}(X).$$

They both coincides with their primitive part, since $H^2_*(X, \mathcal{O}_X) = 0$. In the case of projective hypersurfaces it holds as well

$$H^{n-p,p}_{\text{prim}}(X) \cong (T^1_{A_X})_{(p-1)d-m} \cong H^1(T_X((p-1)d-m).$$

These spaces can be shown to be isomorphic a priori, without deducing it from the previous theorem. In the third chapter we have shown how this is implied by the vanishings of $H^q(\Omega^p_{\mathbb{P}}(k))$ for $p \geq 0, q, k > 0$ by Bott's theorem (and Hard Lefschetz theorem). On the Grassmannian Gr(k,n) the vanishing of the cohomology group of twisted differentials is a more subtle question. Borel-Bott-Weil theorem is the main source to address the computations of this cohomology groups. A classical survey can be found for example in Snow's paper [110]. The following lemma provides of the vanishing required in the Grassmannian case.

Lemma 7.1.4. *Let $X \subset \mathrm{Gr}(k,n)$ a smooth hypersurface of degree d and let $p \in \{1,\ldots,N-2\}$. Suppose that the following vanishing holds*

(I) $H^{p-1}(\Omega_G^{N-p}(d)) = 0$;

(II) $H^{p}(\Omega_G^{N-p}(d)) = 0$;

(III) $H^{p}(\Omega_G^{N-p}) = 0$;

(IV) $H^{p+1}(\Omega_G^{N-p}) = 0$.

Then the following isomorphism holds

$$H^{p-1}(\bigwedge^{p-1} T_X(2d-n)) \cong H^p(\bigwedge^{p} T_X(d-n)).$$

Proof. Consider the tangent-normal sequence raised to the p-th power

$$0 \to \bigwedge^{p} T_X(d-n) \to \bigwedge^{p} TG|_X(d-n) \to \bigwedge^{p-1} T_X(2d-n) \to 0.$$

The long associated sequence in cohomology is

$$\cdots \to H^{p-1}(\bigwedge^{p} TG|_X(d-n)) \to H^{p-1}(\bigwedge^{p-1} T_X(2d-n)) \to H^p(\bigwedge^{p} T_X(d-n)) \to H^p(\bigwedge^{p} TG|_X(d-n)) \to \cdots$$

By Serre duality

$$H^{p-1}(\bigwedge^p TG|_X(d-n)) \cong H^{p-1}(\Omega_G^{N-p}|_X(d));$$

$$H^{p}(\bigwedge^p TG|_X(d-n)) \cong H^{p}(\Omega_G^{N-p}|_X(d)).$$

Using the Koszul complex one has that the vanishing conditions (I,III) imply the of $H^{p-1}(\Omega_G^{N-p}|_X(d))$, and the same with $H^p(\Omega_G^{N-p}|_X(d))$ and conditions (II, IV). $\square$

The above Lemma gives us only one step of the iterated multiplication map. However, one can replicate the same technique and get

Lemma 7.1.5. *Let $X \subset \mathrm{Gr}(k,n)$ a smooth hypersurface of degree d and let $p \in \{1,\dots,N-2\}$. Suppose that the following vanishings hold*

(I) $H^i(\Omega_G^{N-1-i}(jd)) = 0,\ i,j = 1,\dots,p;$

(II) $H^{i+2}(\Omega_G^{N-1-i}((j-1)d)) = 0,\ i,j = 1,\dots,p;$

(III) $H^{i+1}(\Omega_G^{N-1-p}((j-1)d)) = 0,\ i,j = 1,\dots,p;$

(IV) $H^{i+1}(\Omega_G^{N-1-p}(jd)) = 0,\ i,j = 1,\dots,p.$

Then the following isomorphism holds

$$H^1(T_X(pd)) \cong H^p(X, \Omega_X^{N-1-p}).$$

Proof. A diagram-chasing based on the two sequences

$$0 \to \Omega_G^t(c-d) \to \Omega_G^t(c) \to \Omega^t(c)_G|_X \to 0 \tag{7.5}$$

and

$$\Omega_X^{t-1}(c-d) \to \Omega^t(c)_G|_X \to \Omega_X^t(c) \to 0. \tag{7.6}$$

In particular the $k-$th step of the lemma above, that is the isomorphism between

$$H^k(\Omega_X^{n-k}((p-k))d) \to H^{k+1}(\Omega^{n-k-1}((p-k-1)d))$$

is controlled by the vanishings of

$$H^k(\Omega_G^{N-1-k}((p-k)d)) = 0,\ H^{k+1}(\Omega_G^{N-1-}((p-k-1)d)) = 0,\ \text{and}$$

$$H^{k+1}(\Omega_G^{N-1-k}((p-k)d)) = 0,\ H^{k+2}(\Omega_G^{N-1-k}((p-k-1)d)) = 0.$$

□

What we have to understand now is for which $X_d \subset \mathrm{Gr}(k,n)$ the vanishing conditions of lemma 7.1.5 are authomatically satisfied. Borel-Bott-Weyl theorem transform the vanishing question into a combinatorial one. In particular a result by Snow ([110]) is particularly effective in our context.

Theorem 7.1.6 (Thm. 3.2 and 3.4 in [110])**.** *Consider the Grassmannian* $\mathrm{Gr}(k,n)$*, and let* $N := k(n-k)$ *as before,* $t \geq 1$*. Then* $H^p(G, \Omega^q(t)) = 0$ *if any of the following conditions are satisfied.*

(I) $t \geq n$*;*

(II) $kp \geq (k-1)q > 0$;

(III) $p > N - q$*;*

(IV) $q > N - k$*;*

(V) $q \leq t$*;*

The above theorem is particularly effective in our context. Indeed we are now in position to prove the main result of this section. We recall first the description of the Hodge groups of the Grassmannian. As in the projective case, $h^{i,j}(\mathrm{Gr}) = 0$ for $i \neq j$. On the other hand when $i = j$ the dimension of these spaces can be easily computed as

$$h^{j,j}(G) = \#\{(a_1, \ldots, a_k) | n-k \geq a_1 \geq \ldots \geq a_k \geq 0,\ \sum a_i = j\}.$$

We will need the following definition

Definition 7.1.7. Let $G = \mathrm{Gr}(k,n)$. We define $I_{j-1,j}$ as the cokernel of the natural inclusion map

$$0 \to H^{j-1,j-1}(G) \to H^{j,j}(G).$$

Theorem 7.1.8. *Let* X_d *a smooth hypersurface in the Grassmannian* $G = \mathrm{Gr}(k,n)$*. Let* $N = \dim(G) = k(n-k)$*, and* R_f^G *the Jacobian ring for* X *defined in 7.1.2. Assume that* $d \geq n-1$*. If* $\dim(X) = N - 1 \equiv 0\ (2)$*. Then*

$$[R_f^G]_{(p+1)d-n} \cong H^p_{\mathrm{prim}}(X, \Omega^{N-1-p}).$$

If $\dim(X) = N-1 \equiv 1\ (2)$ *then*

$$[R_f^G]_{(p+1)d-n} \cong H^p_{\text{prim}}(X, \Omega^{N-1-p}) \oplus \delta_{p,\frac{N}{2}} I_{p-1,p},$$

where $\delta_{p,\frac{N}{2}}$ *is the Kronecker delta symbol.*

Proof. First we point out that when the dimension of X is odd, then $H^{N-1}(X) = H^{N-1}_{\text{prim}}(X)$.
By Lemma 7.1.3 one has

$$(R_f^G)_{k+d} \cong (T^1_{A_X})_k \cong H^1(X, T_X(k)).$$

In particular, thanks to Lemma 7.1.5 we will have

$$(R_f^G)_{pd-n+d} \cong H^1(X, T_X(pd-n)) \cong H^p(X, \Omega^{N-1-p}),$$

provided that the vanishing conditions (I-IV) and (V) for the index 1 case hold. By Thm. 7.1.6, part I, all these vanishings are automatically satisfied if $d \geq n-1$, except possibly $H^{i+1}(\Omega_G^{N-1-p}) = H^{i+1}(\Omega_G^{N-1-p}) = 0$.
Thanks to the given description of the cohomology ring of the Grassmannian, we know that the above groups vanishes for almost all values of i. In particular from 7.3 and 7.4 one gets the sequence in cohomology

$$0 \to H^{N-1-p,p-1}(G) \to H^{N-p,p}(G) \to H^{p-1}(\Omega_X^{N-p}(d)) \to H^{N-1-p,p}(X) \to H^{N-p,p+1}(G) \to 0,$$

where we have already taken into account all the other vanishings of Lemma 7.1.5. The Hodge groups in the Grassmannian will vanish unless $p+1 = N-p$ or $p = N-p$. In the first case $\dim(X) = N-1 = 2p$ is even, and we have

$$0 \to H^{p-1}(\Omega_X^{N-p}(d)) \to H^{N-1-p,p}(X) \to H^{N-p,p+1}(G) \to 0,$$

that is

$$H^1(T_X((\frac{N-1}{2}d)) \cong H^{\frac{N-3}{2}}(\Omega_X^{\frac{N+1}{2}}(d)) \cong H_{\text{prim}}^{\frac{N-1}{2},\frac{N-1}{2}}(X).$$

In the second case the dimension of X is odd, we have $N = 2p$ and

$$0 \to H^{N-1-p,p-1}(G) \to H^{N-p,p}(G) \to H^{p-1}(\Omega_X^{N-p}(d)) \to H^{N-1-p,p}(X) \to 0,$$

that is

$$H^1(T_X((\frac{N}{2}d)) \cong H^{\frac{N}{2}}(\Omega_X^{\frac{N}{2}}(d)) \cong H^{\frac{N-1}{2},\frac{N}{2}}(X) \oplus I_{\frac{N-2}{2},\frac{N}{2}}$$

□

The above theorem guarantees an extension of the Griffiths Residue calculus to all but a finite number of case of any Grassmannian (namely, in the Fano case of index > 1). Of course Borel-Bott-Weyl theorem can be effectively used to get either more vanishing or to easily compute the exceptions to the above result in the Fano case. As we have seen, in general for a Grassmannian Gr(k,n) the difference between $(R_f)^G_{(p+1)d-n}$ and $H^{p,n-p}_{\text{prim}}(X)$ can be computed in terms of $H^p(\Omega^q_G(k))$. There exists ad-hoc formulas for these group, but a general statement is complicate to find. The situation is slightly better for the Grassmannian of lines Gr(2,n). Here we can use the Peternell-Wisniewski vanishing results, Lemma 0.1 in [91] (basically a re-working of Saito conditions).

Corollary 7.1.9. *Let X a smooth hypersurface of degree d in the Grassmannian Gr(k,n) defined by an $f \in H^0(\mathcal{O}_G(d))$. Then*

$$\bigoplus_{p=1}^{N-1}(R_f)_{(p+1)d-n} \oplus B_{N-1-p,p} \cong \oplus(H^{N-1-p,p}_{\text{prim}}(X) \oplus A_{N-1-p,p})$$

with the possible residual contributions A_{p,N_p-1}, B_{p,N_p-1} determined by the non-vanishing of the groups in 7.1.4.

Corollary 7.1.10. *Let X a smooth hypersurface of degree d in the Grassmannian Gr(2,n) defined by an $f \in H^0(\mathcal{O}_G(d))$ $d \leq n-2$. Then*

$$\bigoplus_{p=1}^{n-1}(R_f)_{(p+1)d-n} \cong \oplus H^{N-1-p,p}_{\text{prim}}(X) \oplus \delta_{p,\frac{N}{2}} I_{p-1,p}$$

with the possible exceptions of

$$p = \frac{2n-1-d}{3} \text{ and } p = \frac{4n-9-d}{3}.$$

Proof. Combine Lemma 7.1.4 with the vanishings in [91], Lemma 0.1. □

7.2 Jacobian rings in practice: explicit Hodge groups for hypersurfaces in Grassmannians

In the next section we see how to make this generators totally explicit in a few handy cases. We focus on the Fano of index > 1, in order to give a concrete example of how to effectively compute the Hodge groups even in presence of residual contributions from the ambient space.

7.2.1 Quadric in Gr(2,5) (Gushel-Mukai type)

Our first concrete example involves a smooth quadric fivefold hypersurface in the Grassmannian Gr(2,5). Recall from the introduction that the Gr(2,5) has dimension six, and it is embedded under the Plücker embedding in $\mathbb{P}^9 = \mathbb{P}(\bigwedge^2 V_5)$: therefore by the structure theorem of Buchsbaum-Eisenbud its homogeneous ideal of relations is given by the submaximal Pfaffians of a skew 5 by 5 matrix. In particular it is immediate to write the five equations as

$$I_G = (x_{3,4}x_{2,5} - x_{2,4}x_{3,5} + x_{2,3}x_{4,5}, x_{3,4}x_{1,5} - x_{1,4}x_{3,5} + x_{1,3}x_{4,5}, x_{2,4}x_{1,5} - x_{1,4}x_{2,5} + x_{1,2}x_{4,5},$$

$$x_{2,3}x_{1,5} - x_{1,3}x_{2,5} + x_{1,2}x_{3,5}, x_{2,3}x_{1,4} - x_{1,3}x_{2,4} + x_{1,2}x_{3,4}).$$

As before, we think of X as defined by the vanishing of an (appropriate) single polynomial f in $H^0(\mathcal{O}_G(2))$. X is a first example of a Gushel-Mukai variety in the sense of [69]. Therefore an explicit computation of its Hodge groups is of particular interest. By adjunction formula one has that $\omega_X \cong \mathcal{O}_X(-3)$. In particular we know straight away that

$$H^{0,5}(X) \cong H^{5,0}(X) \cong H^0(K_X) = 0.$$

Lemma 7.2.1. *Let X as above. The following isomorphisms hold*

- $(R_f^G)_{-1} \cong H^1(T_X(-3)) \cong H^{4,1}(X)$*;*
- $(R_f^G)_1 \cong H^1(T_X(-1)) \cong H^{3,2}(X)$*;*
- $(R_f^G)_3 \cong H^1(T_X(1)) \cong H^{2,3}(X)$*;*
- $(R_f^G)_5 \cong H^1(T_X(3)) \cong H^{1,4}(X)$*.*

Proof. The first isomorphism (and, dually, the last) is already estabilished by the T^1 theory of [52] and the (partial) local duality in [105], since $(R_f^G)_k = (T^1_{A_X})_{-d+k}$. The first thing that we have to prove is therefore

$$H^1(T_X(-1)) \cong H^2(\bigwedge^2 T_X(-3)) \cong H^2(\Omega_X^3),$$

where the last isomorphism is given by Serre duality. For the Grassmannian G=Gr(2,5) the following vanishings hold (cf. [91] or [105])

$$H^1(\Omega_G^4(2)) = H^1(\Omega_G^4) = H^2(\Omega_G^4(2)) = H^3(\Omega_G^4) = 0.$$

In particular Lemma 7.1.4 applies, and so we conclude.
We are left to prove $H^1(T_X(1)) \cong H^2(\bigwedge^2 T_X(-1)) \cong H^3(\bigwedge^3 T_X(-3)) \cong H^{2,3}(X)$: note that since $3 > d$ we cannot conclude immediately by local duality. Equivalently, we have to prove that

$$H^2(\Omega_X^3(2)) \cong H^3(\Omega_X^2).$$

This is done assuming both

$$H^2(\Omega_G^3|_X(2)) = H^3(\Omega_G^2|_X) = 0.$$

Since both $H^3(\Omega_G^2) = H^4(\Omega_G^2(2)) = 0$ from lemma 7.1.4 we have the vanishing of the second group. The first vanishing is a bit trickier. Indeed conditions of Lemma 7.1.4 are not satifisied a priori. What we have is in fact an exact sequence

$$H^2(\Omega_G^3(2)) \to H^2(\Omega_G^3|_X(2)) \to H^3(\Omega_G^3) \to H^3(\Omega_G^3(2))$$

Thanks to vanishings of the first and last term above we have

$$H^2(\Omega_G^3|_X(2)) \cong H^3(\Omega_G^3),$$

the latter being nonzero. Therefore substituting in the normal-exact sequence we have

$$0 \to H^2(\Omega_X^2) \to H^3(\Omega_G^3) \to H^2(\Omega_X^3(2)) \to H^3(\Omega_X^2) \to 0.$$

By Lefschetz hyperplane section theorem $H^2(\Omega_X^2) \cong H^2(\Omega_G^2) \cong H^3(\Omega_G^3)$ both being 2-dimensional, and this concludes the proof. $\square$

Now that we estabilished the isomorphisms in abstract, we want to explicitely compute the Jacobian ring of a Gushel-Mukai fivefold.
We have therefore to make explicit the action of $\mathfrak{sl}_5$ on $H^0(\mathrm{Gr}(2,5), \mathcal{O}_G(2))$, the latter

being the quotient of $\mathbb{C}[x_{1,2},\ldots,x_{4,5}]$ by the ideal generated by the Plücker relations. If we unravel the previous definition we have that D_j^i acts as

$$D_j^i(x_{r,s}\cdot x_{h,k}) = (\delta_{j,r}x_{i,s}+\delta_{j,s}x_{r,i})x_{h,k}+x_{r,s}\cdot(\delta_{j,h}x_{i,k}+\delta_{j,k}x_{h,i}).$$

Extending by linearity we can rewrite everything in a much more neat form as D_j^i as

$$D_j^i = \sum_{k=1}^{5} x_{k,i}\frac{\partial}{\partial x_{k,j}}$$

We prepared a Macaulay2 script that, given a polynomial $f \in H^0(\mathrm{Gr}(2,5),\mathcal{O}_G(2))$ returns the 24 polynomials $D_j^i(f)$. The polynomial f needs to be chosen such that the corresponding X is smooth: in turn this can be checked a posteriori. In fact recall from [108] that a projective variety X is smooth if and only if $T^1_{A_X}$ is finite dimensional, and this property can be easily checked by computer algebra.
In particular the Fermat-type polynomial

$$f=\sum a_{i,j}x_{i,j}^2$$

works as a choice, as long as we take the coefficients $a_{i,j}$ in a fairly generic way. In particular none of the $D_i^j(f)$ has to cancel out and become identically zero: to this purpose picking $a_{i,j}\neq a_{r,s}$ will be enough. As an example with random coefficients we can therefore pick

$$f = x_{1,2}^2+2x_{1,3}^2+4x_{1,4}^2+5x_{1,5}^2+6x_{2,3}^2+11x_{2,4}^2+75x_{2,5}^2+13x_{3,4}^2+43x_{3,5}^2+8x_{4,5}^2.$$

Using the formula above we write the twenty-four differential polynomial as

$$\begin{aligned} D_1^2(f) &= 4x_{1,3}x_{2,3}+8x_{1,4}x_{2,4}+10x_{1,5}x_{2,5}\\ &\vdots\\ D_4^4(f)-D_5^5(f) &= 8x_{1,4}^2-10x_{1,5}^2+22x_{2,4}^2-150x_{2,5}^2+26x_{3,4}^2-86x_{3,5}^2\end{aligned}$$

Denote by D the ideal generated by the 24 polynomials above and f. Let P the ideal generated by the Plücker equation. By the description above we have

$$R_f^G \cong \mathbb{C}[x_{1,2},\ldots,x_{4,5}]/(P+D).$$

We compute the Hilbert-Poincaré series of R_f^G, this being

$$\mathrm{HP}(R_f^G) = 1 + 10t + 25t^2 + 10t^3 + t^4.$$

By Lemma 7.2.1 we have $0 = (R_f^G)_{-1} \cong H^{4,1}(X) \cong \overline{H^{1,4}(X)}$ and $\mathbb{C}^{10} = (R_f^G)_1 \cong H^{3,2}(X) \cong \overline{H^{2,3}(X)} \cong (R_f^G)_3 \cong (R_f^G)_1^\vee$. This coincides with the calculation already done above.

Notice in particular that $(R_f^G)_1 \cong H^{3,2}(X)$ is generated by the degree 1 element in R, that is the ten linear forms $\{x_{i,j}\}$, dual to $H^{2,3}(X) \cong (R_f^G)_3$ with respect to the socle generator $x_{4,5}^4$ of R_4.

7.2.2 Cubic hypersurface in Gr(2,5)

The second example we compute in full details is a smooth cubic hypersurface X_3 in the Grassmannian Gr(2,5). Its behaviour will be different from the quadric case above: indeed we will see how here the duality of the Jacobian ring will be obstructed by some residual cohomology group of Gr(2,5).

First we compute in a purely topological way the Euler characteristic of X.

We use some Pieri-Giambelli techniques from Schubert Calculus. Here we follow the notation of [56]. We write the total Chern class of $Gr(2,5)$ as

$$\begin{pmatrix} 1 & & & \\ 5 & 12 & & \\ 11 & 30 & 25 & \\ 15 & 35 & 30 & 33 \end{pmatrix}$$

(with rows and columns labelled from 0), where the i,j-th element is the coefficient of the Schubert cycles $\sigma_{i,j}$. From the normal exact sequence we have

$$\frac{c(Gr(2,5))}{(1+3\sigma_1)} = c(X),$$

where $\sigma_1 = \sigma_{1,0}$ denotes the (ample) generator of the Picard group.

To compute $c(X)$ we use the Pieri's formula, that is

$$\sigma_{a,b}\sigma_1 = \sigma_{a+1,b} + \sigma_{a,b+1}.$$

In fact, if $c(G) = \sum a_{i,j}\sigma_{i,j}$ and $c(X) = \sum \lambda_{i,j}\sigma_{i,j}$ using the above formula one gets

$$a_{i,j} = \lambda_{i,j} + 3\lambda_{i-1,j} + 3\lambda_{i,j-1}.$$

It is then easy to compute

$$c(X) = \begin{pmatrix} 1 & & & \\ 2 & 6 & & \\ 5 & -3 & 34 & \\ 0 & 44 & -204 & * \end{pmatrix}$$

We have therefore

$$c_5(X) = (-204\sigma_{3,2})3\sigma_1 = -612\sigma_{3,3}$$

and this implies $e(X) = -612$. We proceed now with our analysis of the Jacobian ring. As before

$$H^{0,5}(X) \cong H^{5,0}(X) \cong H^0(K_X) = 0.$$

Lemma 7.2.2. *Let X as above. We will denote by V_5 the five dimensional $\mathbb{C}$-vector space such that* $\mathrm{Gr}(2,5) \cong \mathrm{Gr}(2,V_5)$. *Then we have*

$$\begin{aligned} (R_f^G)_1 &\cong H^1(T_X(-2)) \cong H^{4,1}(X) \\ (R_f^G)_4 &\cong H^1(T_X(1)) \cong H^{3,2}(X) \oplus V_5 \\ (R_f^G)_7 &\cong H^1(T_X(4)) \cong H^{2,3}(X) \\ (R_f^G)_{10} &\cong H^1(T_X(7)) \cong H^{1,4}(X). \end{aligned}$$

Proof. As before, the first and the last isomorphism follows by T^1 theory and Saito-duality. In the case $(R_f^G)_7 \cong H^1(T_X(4)) \cong H^{2,3}(X)$ the conditions of lemma 7.1.4 hold, therefore the conclusion is authomatic. The only case to spell out in full details is

$$(R_f^G)_4 \cong H^1(T_X(1)) \cong H^{3,2}(X) \oplus V_5.$$

Equivalently, we have to prove $H^1(\Omega_X^4(3))/V_5 \cong H^2(\Omega_X^3)$. Using normal-exact sequence we get (since $H^{3,1}(X) = 0$ by Lefschetz theorem on hyperplane section)

$$0 \to H^1(\Omega_G^4|_X(3)) \to H^1(\Omega_X^4(3)) \to H^2(\Omega_X^3) \to H^2(\Omega_G^4|_X(3)).$$

Conditions (2,4) in lemma 7.1.4 assure the vanishing on the last term in the sequence. On the other hand we have that condition (1) does not apply. In particular we have the sequence

$$0 \to \Omega_G^4 \to \Omega_G^4(3) \to \Omega_G^4|_X(3) \to 0$$

and, since $H^1(\Omega_G^4) = H^2(\Omega_G^4) = 0$, the isomorphism

$$H^1(\Omega_G^4|_X(3)) \cong H^1(\Omega_G^4(3)) \cong H^5(\Omega_G^2(-3)) \cong V_5,$$

with the last isomorphism coming from Borel-Bott-Weyl theorem (cf. [84], Proposition 3.3). We have

$$H^1(T_X(1)) \cong H^{3,2}(X) \oplus H^1(\Omega_G^4(3)) \cong H^{3,2}(X) \oplus V_5.$$

$\square$

We compute the Jacobian ring. We follow the same rule of the quadric case. If we pick a (generic) cubic form $f \in H^0(\mathcal{O}_G(3))$ we compute the Hilbert-Poincaré Series of Jacobian ring R_f^G as

$$\mathrm{HP}(R_f^G) = 1+10t+50t^2+150t^3+305t^4+421t^5+421t^6+300t^7+150t^8+50t^9+10t^{10}+t^{11}$$

From the results above it follows

$$(R_f^G)_1 \cong H^{4,1}(X) \cong \mathbb{C}^{10} \cong H^{1,4}(X) \cong (R_f^G)_{10},$$

$$(R_f^G)_4 \cong H^{3,2}(X) \oplus V_5 \cong \mathbb{C}^{300} \oplus \mathbb{C}^5,$$

$$(R_f^G)_7 \cong H^{2,3}(X) \cong \mathbb{C}^{300},$$

and moreover

$$(R_f^G)_3 \cong H^1(T_X) \cong \mathbb{C}^{150}.$$

In particular the Hodge diamond is

$$\begin{array}{ccccccccccc}
0 & & 10 & & 300 & & 300 & & 10 & & 0 \\
 & 0 & & 0 & & 2 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

and this coincides with the Euler characteristic computation above. Explicit generators for the Hodge groups can be easily extracted from the computations above.

7.2.3 Quadric in Gr(2,6)

We consider now an example in a different Grassmannian, that is Gr(2,6). Recall that this Grassmannian is embedded via Plücker in $\mathbb{P}^{14}$. It is a smooth Fano of dimension 8 and canonical class $\omega_{\mathrm{Gr}(2,6)} \cong \mathcal{O}_G(-6)$. Let $X_2 \subset \mathrm{Gr}(2,6)$ be a smooth quadric hypersurface defined by a generic $f \in H^0(\mathrm{Gr}(2,6), \mathcal{O}_G(2))$. This is a Fano manifold of Calabi-Yau type, in the sense of [69]. Here is shown in particular that $h^{5,2} = 1$ and $H^1(T_X) \cong H^{4,3}(X)$. We do the same computations using our graded ring method.

Proposition 7.2.3. *Let $X \subset \mathrm{Gr}(2,6)$ smooth, given by the vanishing of a general f of degree 2 form in $H^0(\mathrm{Gr}(2,6), \mathcal{O}_G(2))$. Let $\delta_{p,4}$ the Kronecker delta symbol. Then*

$$\bigoplus_{p=1}^{n-1} (R_f^G)_{(p+1)d-6} \cong \oplus H^{p,7-p}_{\mathrm{prim}}(X) \oplus \delta_{p,4}\mathbb{C}.$$

Moreover the isomorphism

$$H^1(T_X) \cong H^3(\Omega^4_X)$$

holds.

Proof. We use the condition listed in theorem 7.1.10. Indeed the only possible exceptions for $(R_f^G)_{(p+1)d-5} \cong \oplus H^{p,7-p}_{\mathrm{prim}}(X)$ happens when $p = 4$. To conclude we prove that $H^2(\Omega^4_X(2)) \cong H^4(\Omega^3_X) \oplus \mathbb{C}$. From

$$0 \to \Omega^3_X \to \Omega^4_G|_X(2) \to \Omega^4_X(2) \to 0$$

one has in cohomology

$$0 \to H^{3,3}(X) \to H^3(\Omega^4_G|_X(2)) \to H^3(\Omega^4_X(2)) \to H^{4,3}(X) \to 0.$$

On the other hand by writing the long exact sequence associated in cohomology to

$$0 \to \Omega^4_G \to \Omega^4_G(2) \to \Omega^4_G|_X(2) \to 0$$

one has the isomorphism

$$H^3(\Omega^4_G|_X(2) \cong H^{4,4}(G).$$

The result follows then from Lefschetz's theorem on hyperplane section and the computations of

$$H^{3,3}(G) \cong \mathbb{C}^2; \; H^{4,4}(G) \cong \mathbb{C}^3.$$

The isomorphism

$$H^1(T_X) \cong H^3(\Omega^4_X)$$

follows from the above formula with $p = 2$. □

We can compute the Jacobian ring of the quadric X by proceding as in the examples in the case of $\mathrm{Gr}(2,5)$. Notice that here the Jacobian ideal will be generated by 35 derivations, but the computations remains identical in substance. In particular, chosing a generic f we get as Hilbert-Poincaré series of the Jacobian ring the following

$$\mathrm{HP}(R^G_f) = 1 + 15t + 69t^2 + 112t^3 + 70t^4 + 16t^5 + t^6,$$

as prescribed by proposition 7.2.3. it follows that the Hodge diamond of X is

$$\begin{array}{ccccccccccccccc}
0 & & 0 & & 1 & & 69 & & 69 & & 1 & & 0 & & 0 \\
& 0 & & 0 & & 0 & & 2 & & 0 & & 0 & & 0 & \\
& & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
& & & 0 & & 0 & & 2 & & 0 & & 0 & & & \\
& & & & 0 & & 0 & & 0 & & 0 & & & & \\
& & & & & 0 & & 1 & & 0 & & & & & \\
& & & & & & 0 & & 0 & & & & & & \\
& & & & & & & 1 & & & & & & &
\end{array}$$

Chapter 8

Griffiths residues for complete intersections in Grassmannian

In the previous section we produced a way to associate to an hypersurface X of a Grassmannian a polynomial ring R_f^G whose some special degree components contains the Hodge groups of X.

Here we want to generalise the result to the case of $Z = Z_{d_1,\ldots,d_c} \subset \mathrm{Gr}(k, n)$, a smooth complete intersection (of multidegree $d_1, \ldots, d_c$). We will associate to Z a bi-graded ring $\mathcal{U}_{a,b}$ that in appropriate bi-degree components, will give us back the (central) Hodge groups of Z. We will call such a ring a *Griffiths ring*: we will use another letter different from R to differentiate it from the hypersurface case. As before, the results could be adapted with minor modification to the more general setting of homogeneous spaces. However, we focus here only on the Grassmannian case, leaving the rest for future works.

The idea follows from the Cayley trick approach of Dimca, Konno, Terasoma et al. in the projective case. Starting from Z we will construct a hypersurface $\widehat{Z}$ in a projective bundle $\mathbb{P}(\mathcal{E})$ over $\mathrm{Gr}(k, n)$. The cohomology of Z and of $\widehat{Z}$ coincides up to a shift. Therefore we can apply the Jacobian-like construction of the previous section to $\widehat{Z} \subset \mathbb{P}(\mathcal{E})$, with suitable modifications. Here and in the next pages, we will often refer to the work of Konno, [77]. In particular in the cited paper, a generalised version of a Griffiths ring for a variety defined by the zero set of a generic section of $\mathcal{E}$ is defined. However the the result was made explicit only for complete intersections in a projective space. In what we consider being the main result of this chapter, we give an explicit version of the Griffiths residue theorem for complete intersections in Grassmannians as well. We will now go through a recap on projective bundles on a arbitrary smooth projective variety

X

Projective bundles and their cohomology Let $\mathcal{E}$ an holomorphic vector bundle of rank c (or the locally free sheaf of its sections) on an n dimensional compact manifold X. Denote by $\mathcal{E}_x$ the fiber over $x \in X$. Consider the projective vector bundle

$$\pi : Y = \mathbb{P}(\mathcal{E}) \to X$$

whose fiber is the space $\mathbb{P}(\mathcal{E}_x)$. Useful sequences to understand the geometry of Y in terms of X are the *relative tangent sequence*

$$0 \to T_{Y/X} \to T_Y \to \pi^* T_X \to 0 \tag{8.1}$$

and the *relative Euler sequence*

$$0 \to \mathcal{O}_Y \to \pi^* \mathcal{E}^* \otimes \mathcal{L} \to T_{Y/X} \to 0, \tag{8.2}$$

where $\mathcal{L} = \mathcal{O}_Y(1)$ denotes the (ample) dual of the tautological line bundle on the projective bundle Y. The following lemma is extremely useful in this context.

Lemma 8.0.1 (Lemma 1.2, [77]). *Let V be any holomorphic vector bundle on X. Then*

$$H^q(Y, \pi^* V \otimes \mathcal{L}^h) \cong \begin{cases} H^q(X, V \otimes \mathrm{Sym}^h \mathcal{E}) & \textit{if } h \geq 0 \\ H^{q-c+1}(X, V \otimes det\ \mathcal{E}^* \otimes \mathrm{Sym}^{-h-c} \mathcal{E}^*) & \textit{if } h \leq -c \\ 0 \quad \textit{otherwise} \end{cases}$$

From the above Lemma it follows that $H^0(X, \mathcal{E}) \cong H^0(Y, \mathcal{L})$. Explicitely, take U an open subset of X over which $\mathcal{E}$ is trivial and let $e_1, \dots, e_r$ a frame of $\mathcal{E}$ on U. If $\sigma \in H^0(X, \mathcal{E})$ is given locally by $\sigma = \sum \sigma_i e_i$, then the section $\hat{\sigma} = \sum_i \sigma_i e_i$, where we regard e's as homogeneous fiber coordinates on $\mathbb{P}(\mathcal{E}) \cong U \times \mathbb{P}^{c-1}$. Let Z and $\widehat{Z}$ the zero varieties of σ and $\hat{\sigma}$. Note that $\widehat{Z} \in H^0(Y, \mathcal{L})$. The Hodge theory of Z and $\widehat{Z}$ are strongly related: namely we have the following result

Proposition 8.0.2 (Proposition 4.3, [77]). *There is a canonical isomorphism of Hodge structures*

$$H^q_{\mathrm{van}}(Z, \mathbb{C})(1-c) \cong H^{q+2c-2}_{\mathrm{van}}(\widehat{Z}, \mathbb{C}).$$

Moreover, since for X a complete intersection in G, and G itself has no primitve cohomology, then on Z primitive and vanishing cohomology agrees, and we will use here the primitive notation, mostly for historical reasons. The Hodge theory of $\widehat{Z}$ can be now described in terms of the generalised Jacobian ring (or pseudo-Jacobian system) of Mark Green. Our definition will differ slightly from the classical one, in order to further simplify the computations. Moreover we explicitely describe what we called the Griffiths ring for complete intersections in Grassmannians.

8.1 Griffiths ring for complete intersections in Grassmannians

Let us now fix some notation. As before $Gr(k,n)$ will denote the Grassmannian of k-planes in a fixed n-dimensional $\mathbb{C}$-vector space V_n. If there is no danger of confusion, we will denote the Grassmannians by G. The dimension of the Grassmannian is $N = k(n-k)$. Let $Z = Z_{d_1,\dots,d_c}$ a smooth codimension c complete intersection in the Grassmannian of multi-degree $d_1, \dots, d_c$. We will call $m = \sum d_i - n$ the adjunction degree of Z: in particular $\omega_Z \cong \mathcal{O}_Z(m)$. Z is defined by a section $\sigma \in H^0(G, \mathcal{E})$, where $\mathcal{E} = \bigoplus^c \mathcal{O}_G(d_i)$. We associate to Z an hypersurface $\widehat{Z} \subset Y = \mathbb{P}(\mathcal{E})$ as explained above. We denote by $\widehat{N} = N + (c-1)$ the dimension of Y. The projective bundle Y has $\mathrm{Pic}(Y) \cong \mathbb{Z}^2$: pick as a $\mathbb{Z}$-basis $\langle \mathcal{L}, D \rangle$ with $D = \pi^* \mathcal{O}_G(1)$. With respect to this grading, we write $\mathcal{F}(a,b) := \mathcal{F} \otimes \mathcal{L}^a \otimes D^b$ and $H^i_{*,*}(\mathcal{F})$ for $\bigoplus_{a,b} H^i(\mathcal{F}(a,b))$. We define the *Griffiths ring* of Z as follows

Definition 8.1.1. Let $Z, \widehat{Z}$ as above. The Griffiths ring of Z is

$$\mathcal{U} = \bigoplus_{a,b} \mathcal{U}_{a,b}$$

with

$$\mathcal{U}_{a,b} = H^1(\widehat{Z}, T_{\widehat{Z}} \otimes \mathcal{L}^{a-1} \otimes D^b). \tag{8.3}$$

Notice that a priori $\mathcal{U}$ above has only the structure of (bi)-graded vector space. The ring structure is given by the following tangent-normal exact sequence (denoting with an abuse of notation with $\mathcal{L}$ as well the restriction of $\mathcal{L}$ to $\widehat{Z}$

$$0 \to T_{\widehat{Z}} \to T_Y|_{\widehat{Z}} \to \mathcal{O}_{\widehat{Z}}(\mathcal{L}) \to 0$$

For any $(a-1,b)$ we consider the twisted version of the above sequence

$$0 \to T_{\widehat{Z}}(a-1,b) \to T_Y|_{\widehat{Z}}(a-1,b) \xrightarrow{\varphi} \mathcal{O}_{\widehat{Z}}(a,b) \to 0.$$

From the sequence 8.1 one has $H^1(T_Y|_{\widehat{Z}}(a,b)) \cong 0$. Therefore, if F denotes the equation of $\widehat{Z}$ one has

$$H^1(\widehat{Z}, T_{\widehat{Z}} \otimes \mathcal{L}^{a-1} \otimes D^b) \cong H^0(Y, \mathcal{L}^a \otimes D^b)/(F, \mathrm{Im}(\varphi)).$$

Therefore the structure of ring of $\mathcal{U}$ descends directly by the one of $H^0_{*,*}(Y, \mathcal{L}^a \otimes D^b)$. We identify

$$\bigoplus_{a,b} H^0(Y, \mathcal{L}^a \otimes D^b) \cong S[y_1, \ldots, y_c]$$

where S denotes the coordinate ring of the affine cone over Grassmannian Gr(k,n). We assign to the Plücker variables x_I the bi-degree (0,1), and to the new 'fiber' variables y_i the bi-degree $(1, -d_i)$. In this set of coordinates F is defined by

$$F = \sum_i y_i f_i,$$

where f_i are the equations of the complete intersection Z. Notice that $F \in (S[y_1, \ldots, y_c])_{1,0}$. A possible alternative interpretation for the bi-grading is to consider it as as coming from the Cox ring of Y. From the relative tangent sequence 8.1 we have that the action of $H^0_{*,*}(T_Y)$ splits into the direct sum of its vertical part and the horizontal part: from the discussion in the previous chapter and Lemma 2.5, [77] we make explicit this action and give a new definition of the Griffiths Ring of Z, that coincides with the one given above.

Definition 8.1.2. Let Z and S as above, with the variables x_I with bi-degree (0,1), and the variables y_i bi-degree $(1, -d_i)$. The *Griffiths ring* of Z can be equivalently defined as

$$\mathcal{U} := S[y_1, \ldots, y_c]/(F, \frac{\partial F}{\partial y_1}, \ldots, \frac{\partial F}{\partial y_c}, \{D_{x_I}(F)\}). \tag{8.4}$$

The derivations D_{x_I} are the ones already defined in the previous chapter. Notice that

$$D_{x_I}(F) = \sum_i y_i D_{x_I}(f_i)$$

and

$$\frac{\partial F}{\partial y_c} = f_i.$$

In turn the above definition can be further simplified as

$$\mathcal{U} := S[y_1, \ldots, y_c]/(F, f_1, \ldots, f_c, \{D_{x_I}(F)\}).$$

From the relative Euler sequence we have $\omega_Y \cong \mathcal{L}^{-c} \otimes D^m$, and by adjunction formula

$$\omega_{\widehat{Z}} \cong \mathcal{L}^{-c+1} \otimes D^m.$$

From the definition 8.1.1 and Proposition 8.0.2 we have the following immediate corollary

Corollary 8.1.3. *$\mathcal{U}_{1,0} \cong H^1(\widehat{Z}, T_{\widehat{Z}}) \cong H^1(Z, T_Z)$.*

We are now able to prove the main result of this chapter. Define δ and I as in 7.1.8.

Theorem 8.1.4. *Let $Z = Z_{d_1,\dots,d_c}$ a smooth complete intersection in a Grassmannian Gr(k,n), and let $\mathcal{U}$ the Griffiths ring attached to Z. Denote by m the canonical degree of Z, that is $\omega_Z \cong \mathcal{O}_Z(m)$. Suppose $m \geq n-1$. Then if $\dim(Z) = N-c$ is even*

$$\mathcal{U}_{p,m} \cong H^{N-c-p,p}_{\mathrm{prim}}(Z).$$

If $\dim(Z) = N-c$ is odd

$$\mathcal{U}_{p,m} \cong H^{N-c-p,p}_{\mathrm{prim}}(Z) \oplus \delta_{p,\frac{N-c}{2}} I_{p,p-1}(G).$$

Proof. The first step is reduce our analysis to the study of Y. From 8.0.2, it is enough to prove that

$$\mathcal{U}_{p+1-c,m} \cong H^{\widehat{N}-1-p,p}_{\mathrm{prim}}(\widehat{Z}).$$

Indeed

$$H^{\widehat{N}-1-p,p}_{\mathrm{prim}}(\widehat{Z}) \cong H^{N+c-2-p,p}_{\mathrm{prim}}(\widehat{Z}) \cong H^{N-p-1,p-c+1}_{\mathrm{prim}}(Z),$$

and relabelling $p' = p+1-c$ we obtain the statement.
By definition of Griffiths ring, we have therefore to show that

$$\mathcal{U}_{p+1-c,m} \cong H^1(\widehat{Z}, T_{\widehat{Z}} \otimes \omega_{\widehat{Z}} \otimes \mathcal{L}^{p-1}) \cong H^1(\widehat{Z}, \Omega^{\widehat{N}-2} \otimes \mathcal{L}^{p-1}) \cong H^p_{\mathrm{prim}}(\widehat{Z}, \Omega^{\widehat{N}-1-p}_{\widehat{Z}}).$$

The only non-obvious isomorphism is the second one. This is proved inductively as follows. First use the two exact sequences (residues and tangent-normal)

$$0 \to \Omega^{k-1}_{\widehat{Z}} \otimes \mathcal{L}^{p-1} \to \Omega^k_{Y|_{\widehat{Z}}} \otimes \mathcal{L}^p \to \Omega^k_{\widehat{Z}} \otimes \mathcal{L}^p \to 0 \tag{8.5}$$

$$0 \to \Omega^k_Y \otimes \mathcal{L}^{p-1} \to \Omega^k_Y \otimes \mathcal{L}^p \to \Omega^k_{Y|_{\widehat{Z}}} \otimes \mathcal{L}^p \to 0 \tag{8.6}$$

From Lemma 4.9 [77], the groups $H^i(Y, \Omega^k_j \otimes \mathcal{L}^{p-1})$ vanishes if $H^r(G, \Omega^s \otimes \det\mathcal{E} \otimes$

$\mathrm{Sym}^t\,\mathcal{E}) = 0$, for specific values of r, s, k. But from 7.1.6 all these groups vanishes when $d \geq n-1$. As in the hypersurface case, the only vanishings that are not automatic are for $H^{p,p}(Y)$. Indeed, chasing the diagram exactly as in 7.1.8 one gets the division in even and odd case. Moreover by Künneth formula, $I_{p,p-1}(Y) = I_{p,p-1}(G)$.
When these vanishings are not satisfied, the residual contributions depends only on $H^*(\Omega^k_Y \otimes \mathcal{L}^j)$. These cohomology groups can be expressed in terms of (cohomology of) $\pi^*\Omega^k_G$ and relative cotangent $\Omega^{c-1}_{Y/G}$ by picking appropriate exterior power of the short exact sequence

$$0 \to \pi^*\Omega^1_G \otimes \mathcal{L}^p \to \Omega^1_Y \otimes \mathcal{L}^p \to \Omega^1_{Y/G} \to 0. \tag{8.7}$$

Equivalently, as in Lemma 1.4, [77], one could use the following spectral sequence

$$E_1^{i,j-1} = H^j(Y, \Omega^{p-i}_{Y/G} \otimes \mathcal{L}^p \otimes \pi^*(\Omega^i_G \otimes V)) \Rightarrow H^j(Y, \Omega^p_Y \otimes \mathcal{L}^p \otimes \pi^* V).$$

The last step is to express the cohomology groups of the exterior power of the relative cotangent in terms of the cohomology groups over the Grassmannian. This is done via the following sequence (sequence (3) in [77])

$$0 \to \Omega^l_{Y/G} \otimes \mathcal{L}^p \otimes \pi^*\Omega^{k-1}_G \to \pi^*(\bigwedge^l \mathcal{E} \otimes \Omega^{k-1}_G) \otimes \mathcal{L}^{p-l} \to \Omega^{l-1} \otimes \mathcal{L}^p \otimes \pi^*\Omega^{k-1}_G \to 0. \tag{8.8}$$

Since Z is a complete intersection in Gr(k,n), its normal bundle in the Grassmannian is $\mathcal{E} = \oplus \mathcal{O}_G(d_i)$. Therefore we are in the situation of lemma 8.0.1, and we can express any cohomology group of the form $H^q(Y, \pi^*\Omega^k \otimes \mathcal{L}^p)$ as a function of either $H^q(G, \Omega^k_G \otimes \mathrm{Sym}^p\,\mathcal{E})$ or $H^{q-r+1}(G, \Omega^k_G \otimes det(\mathcal{E}^*) \otimes \mathrm{Sym}^{-p-c}\,\mathcal{E}^*)$, with both $S^p\mathcal{E}$ and $det(\mathcal{E}^*)$ equals to (the sum of some) $\mathcal{O}_G(d_i)$. $\square$

From the proof of the above theorem we can immediately conclude the following corollary.

Corollary 8.1.5. *Let $Z = Z_{d_1,\dots,d_c}$ a smooth complete intersection in a Grassmannian Gr(k,n), and let $\mathcal{U}$ the Griffiths ring attached to Z. Denote by m the canonical degree of Z, that is $\omega_Z \cong \mathcal{O}_Z(m)$. Then*

$$\mathcal{U}_{p,m} \oplus B_{N-c-p,p} \cong H^{N-c-p,p}_{\mathrm{prim}}(Z) \oplus A_{N-c-p,p},$$

where $A_{N-c-p,p}, B_{N-c-p,p}$ depends only on the residual cohomology groups $H^i(G, \Omega^j_G(k))$ for appropriate values of i, j, k.

We analyse now in full details one example in which actually the residual contributes are not all zero, showing how it is possible to get explicit results without restriction on the degrees.

8.2 Examples and computations

8.2.1 Griffiths ring for a Gushel-Mukai fourfold

We continue with our analysis of the Gushel-Mukai type varieties. We focus now on the GM-fourfold case. This is a smooth complete intersection $Z_{2,1} \subset \mathrm{Gr}(2,5)$. In particular it has dimension 4 and canonical class $\omega_Z(-2)$. To Z is associated the adjoint 6-fold hypersurface $\widehat{Z} \subset \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_G(1) \oplus \mathcal{O}_G(2)$. We want to explicitely compute the residual contribute $A_{p,N-p-1}$ and give an explicit presentation for the Griffiths ring $\mathcal{U}$ associated to Z. The main result here is the following

Proposition 8.2.1. *Let $Z, \widehat{Z}$ as above. We have the following*

- $H^0(\Omega^4_Z) \cong H^1(\Omega^5_{\widehat{Z}}) \cong \mathcal{U}_{0,-2}$;
- $H^1(\Omega^3_Z) \cong H^2(\Omega^4_{\widehat{Z}}) \cong \mathcal{U}_{1,-2}$;
- $H^2_{\mathrm{prim}}(\Omega^2_Z) \cong H^3_{\mathrm{prim}}(\Omega^3_{\widehat{Z}}) \cong \mathcal{U}_{2,-2}/V_5$;
- $H^3(\Omega^1_Z) \cong H^4(\Omega^2_{\widehat{Z}}) \cong \mathcal{U}_{3,-2}$;
- $H^4(\mathcal{O}_Z) \cong H^5(\Omega^1_{\widehat{Z}}) \cong \mathcal{U}_{4,-2}$;

Proof. The first of any row of isomorphisms follows from 8.0.2. We will prove only the first 3 points, the other being analogous and following by duality. Moreover (1) is obvious, since all three terms are equal to zero. So we are left to prove here part (2) and (3). For sake of clarity, we will divide the proof in three separate lemmata.

Lemma 8.2.2. $H^1(T_{\widehat{Z}} \otimes \omega_{\widehat{Z}} \otimes \mathcal{L}) \cong H^2(\Omega^4_{\widehat{Z}})$.

Proof. Before starting, notice that the above lemma proves point (2), since

$$H^1(T_{\widehat{Z}} \otimes \omega_{\widehat{Z}} \otimes \mathcal{L}) =: \mathcal{U}_{1,-2}.$$

By tangent pairing,

$$H^1(T_{\widehat{Z}} \otimes \omega_{\widehat{Z}} \otimes \mathcal{L}) \cong H^1(\Omega^5_{\widehat{Z}} \otimes \mathcal{L}).$$

We use the sequence (8.5) with $k = 5$ and $p = 1$. In cohomology this becomes

$$0 \to H^1(\Omega^5_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) \to H^1(\Omega^5_{\widehat{Z}} \otimes \mathcal{L}) \to H^2(\Omega^4_{\widehat{Z}}) \to H^2(\Omega^5_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) \to 0,$$

with the first and last zeroes given, respectively, by Künneth formula and by Akizuki-Kodaira-Nakano vanishing. Using the same arguments, from sequence (8.6) we immediately get

$$0 \to H^1(\Omega^5_Y \otimes \mathcal{L}) \to H^1(\Omega^5_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) \to 0$$

and

$$0 \to H^2(\Omega^5_Y \otimes \mathcal{L}) \to H^2(\Omega^5_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) \to 0.$$

Consider now sequence (8.7). Since the normal bundle to $\widehat{Z}$ has rank 2, the relative cotangent bundle $\Omega^1_{Y/G}$ is a rank 1 bundle. Therefore the raised relative tangent sequence, when tensored with $\mathcal{L}$ has a particularly simple form

$$0 \to \pi^*\Omega^5_G \otimes \mathcal{L} \to \Omega^5_Y \otimes \mathcal{L} \to \pi^*\Omega^4_G \otimes \Omega^1_{Y/G} \otimes \mathcal{L} \to 0.$$

By 8.0.2,

$$H^i(Y, \pi^*\Omega^5_G \otimes \mathcal{L}) \cong H^i(G, \Omega^5_G(1)) \oplus H^i(G, \Omega^5_G(2)).$$

These groups are all 0 for $i = 1, 2, 3$ (see [91], Lemma 0.1). Therefore

$$H^i(\Omega^5_Y \otimes \mathcal{L}) \cong H^i(\pi^*\Omega^4_G \otimes \Omega^1_{Y/G} \otimes \mathcal{L}), \quad i = 1, 2.$$

Finally, by sequence 8.8

$$0 \to \Omega^1_{Y/G} \otimes \mathcal{L} \otimes \pi^*\Omega^4_G \to \pi^*(\Omega^4_G(2) \oplus \Omega^4_G(1)) \to \mathcal{L} \otimes \pi^*\Omega^4_G \to 0.$$

Using Lemma 8.0.2, Kodaira vanishing and the Peternell-Wisniewski Lemma we have

$$H^j(\pi^*(\Omega^4_G(2) \oplus \Omega^4_G(1))) = 0, \quad j = 0, 1, 2$$

and

$$H^l(\pi^*\Omega^4_G \otimes \mathcal{L}) = 0, \quad l = 1, 2.$$

In particular from all these vanishings

$$H^1(\Omega^5_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) = H^2(\Omega^5_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) = 0$$

and the result follows. $\square$

To prove part (3) of the proposition, we need to combine the two following results.

Lemma 8.2.3. $H^2(\Omega^4_{\widehat{Z}} \otimes \mathcal{L}) \cong H^3_{\text{prim}}(\Omega^3_{\widehat{Z}})$

Lemma 8.2.4. $H^1(\Omega^5_{\widehat{Z}} \otimes \mathcal{L}^2) \cong H^2(\Omega^4_{\widehat{Z}} \otimes \mathcal{L}) \oplus V_5$

The two lemma above together proves the results, since

$$\mathcal{U}_{2,-2} = H^1(T_{\widehat{Z}} \otimes \omega_{\widehat{Z}} \otimes \mathcal{L}^2) \cong H^2(\Omega^4_{\widehat{Z}} \otimes \mathcal{L}) \oplus V_5 \cong H^3_{\text{prim}}(\Omega^3_{\widehat{Z}}),$$

as requested.

Proof of Lemma 8.2.3. We use the same tools of the previous Lemma. The first step is the reduction to

$$0 \to H^2(\Omega^4_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) \to H^2(\Omega^4_{\widehat{Z}} \otimes \mathcal{L}) \to H^3(\Omega^3_{\widehat{Z}}) \to H^3(\Omega^4_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) \to 0$$

Then, since by Künneth formula $H^3(\Omega^4_Y) = 0$ we consider the two induced sequences

$$0 \to H^2(\Omega^4_Y \otimes \mathcal{L}) \to H^2(\Omega^4_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) \to 0,$$

$$0 \to H^3(\Omega^4_Y \otimes \mathcal{L}) \to H^3(\Omega^4_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) \to H^4(\Omega^4_Y) \to 0.$$

From sequences 8.7, 8.8 we get the vanishings of $H^2(\Omega^4_Y \otimes \mathcal{L})$ and $H^3(\Omega^4_Y \otimes \mathcal{L})$. This implies

$$0 \to H^2(\Omega^4_{\widehat{Z}} \otimes \mathcal{L}) \to H^3(\Omega^3_{\widehat{Z}}) \to H^4(\Omega^4_Y) \to 0,$$

and therefore by definition and Lesfchetz hyperplane section theorem

$$H^2(\Omega^4_{\widehat{Z}} \otimes \mathcal{L}) \cong H^3_{\text{prim}}(\Omega^3_{\widehat{Z}}).$$

The contribution of $H^4(\Omega^4_Y)$ can be easily computed from the Künneth formula: in fact

$$H^4(\Omega^4_Y) \cong H^4(Y, \mathbb{C}) \cong H^4(\text{Gr}(2,5)) \otimes H^0(\mathbb{P}^1) \oplus H^3(\text{Gr}(2,5)) \otimes H^1(\mathbb{P}^1) \oplus H^2(\text{Gr}(2,5)) \otimes H^2(\mathbb{P}^1).$$

In particular $H^3(\text{Gr}(2,5)) \cong \mathbb{C}^3$ and $H^2(\text{Gr}(2,5)) \cong \mathbb{C}^2$ and therefore $H^4(\Omega^4_Y) \cong \mathbb{C}^5$.

$\square$

Proof of Lemma 8.2.4. The first thing that we need to show is $H^1(\Omega^4_{\widehat{Z}} \otimes \mathcal{L}) = 0$. By using sequences 8.5 and 8.6 this is indeed equivalent to show $H^1(\Omega^4_Y \otimes \mathcal{L}) = 0$. By sequences 8.7, 8.8 this is granted since $H^0(\Omega^3_G(1)) = H^0(\Omega^3_G(2)) = 0$, see [91]. Therefore we have

$$0 \to H^1(\Omega^5_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) \to H^1(\Omega^5_{\widehat{Z}} \otimes \mathcal{L}^2) \to H^2(\Omega^4_{\widehat{Z}} \otimes \mathcal{L}) \to H^1(\Omega^5_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) \to 0. \qquad (8.9)$$

On the other hand from the residue sequence 8.6

$$H^1(\Omega^5_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) \cong H^1(\Omega^5_Y \otimes \mathcal{L}^2), \quad H^2(\Omega^5_{Y|_{\widehat{Z}}} \otimes \mathcal{L}) \cong H^2(\Omega^5_Y \otimes \mathcal{L}^2).$$

Call $\mathcal{M} = \pi^*\Omega^4_G \otimes \Omega^1_{Y/G} \otimes \mathcal{L}^2$. We see from sequence 8.7

$$H^1(\Omega^5_Y \otimes \mathcal{L}^2) \cong H^1(\mathcal{M}), \quad H^2(\Omega^5_Y \otimes \mathcal{L}^2) \cong H^2(\mathcal{M}).$$

From 8.9 we have

$$H^1(\Omega^5_{\widehat{Z}} \otimes \mathcal{L}^2) \cong H^2(\Omega^4_{\widehat{Z}} \otimes \mathcal{L}) \oplus H^1(\mathcal{M})/H^2(\mathcal{M}).$$

Lemma 1.5, ii, [77] gives $H^0(\mathcal{M}) = 0$. By Borel-Bott-Weil

$$H^0(\pi^*(\Omega^4_G(1) \oplus \Omega^4_G(2)) \otimes \mathcal{L}) \cong H^0(\mathcal{L}^2 \otimes \pi^*\Omega^4_G) \cong \mathcal{V}^2,$$

with the latter denoting the unique irreducible SL(5)-module of highest weight -2. Moreover

$$H^1(\pi^*(\Omega^4_G(1) \oplus \Omega^4_G(2)) \otimes \mathcal{L}) \cong V_5 \oplus V_5 \cong \mathbb{C}^{10}$$

and

$$H^1(\pi^*\Omega^4_G \otimes \mathcal{L}^2) \cong V_5.$$

Therefore by sequence 8.8 $H^1(\mathcal{M})/H^2(\mathcal{M}) \cong V_5$, proving the lemma. □

□

We now construct explicitely the Griffiths ring $\mathcal{U}$. The ambient ring $S[y_1, y_2]$ is the Plücker ring already costructed in the previous section with the two new variables y_1, y_2 added. The variables $x_{i,j}$ have bidegree $(0,1)$ while y_1 and y_2 have bidegree (respectively) $(1,-1)$ and $(1,-2)$. As quadric we choose the same one of the hypersurface case, that is

$$f_2 = x_{1,2}^2 + 2x_{1,3}^2 + 4x_{1,4}^2 + 5x_{1,5}^2 + 6x_{2,3}^2 + 11x_{2,4}^2 + 75x_{2,5}^2 + 13x_{3,4}^2 + 8x_{4,5}^2 + 43x_{3,5}^2$$

while as linear equation we pick

$$f_1 = x_{1,2} + x_{3,4}.$$

Note that the latter is smooth only when $n \leq 5$. The 24 derivations are obtained easily from the formula $\sum y_i D_x(f_i)$, given that we already know how each of the infinitesimal derivations in $D_x(f_i)$ acts from the hypersurface example. For example

$$\begin{aligned} D_1^2(F) &= y_1(4x_{1,3}x_{2,3} + 8x_{1,4}x_{2,4} + 10x_{1,5}x_{2,5}) \\ &\vdots \\ D_4^4(F) - D_5^5(F) &= y_1(8x_{1,4}^2 - 10x_{1,5}^2 + 22x_{2,4}^2 - 150x_{2,5}^2 + 26x_{3,4}^2 - 86x_{3,5}^2) + y_0(x_{3,4}) \end{aligned}$$

Denote by D the ideal generated by all these derivations. We have

$$\mathcal{U} = S[y_1, y_2]/(D, F, f_1, f_2).$$

We compute some of the graded components of $\mathcal{U}$

a/b	-4	-3	-2	-1	0	1	2	3
-1	0	0	0	0	0	0	0	...
0	0	0	0	0	1	10	50	...
1	0	0	1	10	24	10	1	0
2	1	10	25	10	1	0	0	0
3	25	11	1	0	0	0	0	0
4	2	0	0	0	0	0	0	0

We have in particular $\mathcal{U}_{1,0} \cong H^1(T_Z)$, $\mathcal{U}_{-1,-2} = H^{4,0}(Z)$, $\mathcal{U}_{1,-2} \cong H^{3,1}(Z)$, $\mathcal{U}_{2,2} = V_5 \oplus H^{2,2}_{\text{prim}}(Z)$, $\mathcal{U}_{3,-2} = H^{1,3}(Z)$ and $\mathcal{U}_{4,-2} = H^{0,4}(Z)$. In particular the Hodge diamond of $Z = Z_{2,1}$

$$\begin{array}{ccccccccc} 0 & & 1 & & 22 & & 1 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

coincides with the Kuznetsov-Debarre computation.

8.2.2 Again on $X_{1^7} \subset \mathrm{Gr}(2,7)$

The last example we want to describe in details is the Calabi-Yau threefold $X_{1^7} \subset \mathrm{Gr}(2,7)$ already described in details in chapter 6. Since its canonical class is non-negative, theorem 8.1.4 applies directly. In particular its Griffiths ring contains the Hodge groups as special homogeneous slices, without any residual contribution from the ambient Grassmannian. We can pick as seven general equations

$$
\begin{aligned}
f_1 &= x_{1,2} + 2x_{2,6} + 3x_{3,5}\\
f_2 &= x_{1,6} + 4x_{2,5} + 5x_{3,4}\\
f_3 &= x_{1,5} + 6x_{2,4} + 7x_{6,7}\\
f_4 &= x_{1,4} + 8x_{2,3} + 9x_{5,7}\\
f_5 &= x_{1,3} + 10x_{4,7} + 11x_{5,6}\\
f_6 &= x_{1,2} + 12x_{3,7} + 13x_{4,6}\\
f_7 &= x_{3,6} + x_{2,7} + x_{4,5}.
\end{aligned}
$$

Denote by I the ideal generated by this seven equations in the coordinate ring S of the Grassmannian Gr(2,7). We already checked in chapter 6 that the variety defined by this set of equations is smooth. Of course the choice of coefficients is ininfluent, provided that they are sufficiently general. The action of $\mathfrak{sl}_7$ on the coordinate ring of X_{1^7} it is generated by 48 homogeneous degree 1 equation that are easy to write down. So, if as before we denote by

$$F = \sum_{i=1}^{7} y_i F_i,$$

where each y_i has bidegreee $(1,-1)$ we have that

$$\mathcal{U} \cong S[y_1,\dots,y_7]/(D+F+I).$$

The ideal D is generated by the induced $\mathfrak{sl}_7$ action on the ring $S[y_1,\dots,y_7]$. We can easily compute the generators as

$$
\begin{aligned}
D_1^1(F) - D_2^2(F) &= -2x_{2,6}y_1 - 4x_{2,5}y_2 + x_{1,6}y_2 - 6x_{2,4}y_3 + x_{1,5}y_3 - 8x_{2,3}y_4 + x_{1,4}y_4 + x_{1,3}y_5\\
&\vdots\\
D_7^6(F) &= 2x_{2,7}y_1 + x_{1,7}y_2 + 11x_{5,7}y_5 + 13x_{4,7}y_6.
\end{aligned}
$$

We compute then the first (graded) component of the Griffiths ring. These are

a/b	-4	-3	-2	-1	0	1	2	3
-1	0	0	0	0	0	0	0	...
0	0	0	0	0	1	14	70	210
1	0	0	0	7	50	91	28	0
2	0	0	28	84	51	7	0	0
3	0	84	77	14	1	0	0	0
4	210	21	0	0	0	0	0	0

The (vertical) slice with $b = 0$ corresponds to the Hodge groups of X_{1^7}. In fact, in accordance with theorem 8.1.4 we have

$$\begin{aligned}
H^{3,0}(X) &\cong \mathcal{U}_{0,0} \cong \mathbb{C} \\
H^{2,1}(X) &\cong \mathcal{U}_{1,0} \cong \mathbb{C}^{50} \\
H^{1,2}(X) \oplus I_{2,1} &\cong \mathcal{U}_{2,0} \cong \mathbb{C}^{50} \oplus \mathbb{C} \\
H^{0,3}(X) &\cong \mathcal{U}_{3,0} \cong \mathbb{C}.
\end{aligned}$$

It would be interesting to use this method we developed to help solving a various range of problems. In the next and final chapter we give some example of ideas that we plan to develop in the near future.

Chapter 9

Fano varieties of K3 type, conjectures and future directions of research

We list here some of the problems we want to work on in the near future. They have all been inspired by the work on this thesis, and they are currently at various stage of completion.

New surfaces of general type. We believe that the analysis in chapter 6 is still at its very beginning. Open questions include a $\mathbb{Z}/14$ action on S_{42} and all possible constructions coming from the table 6.1.

Further generalisation of the Griffiths ring. Several generalisations may be extracted from chapter 8. Indeed one could easily write down an equivalent version for other ambient varieties, such as the other homogeneous spaces. We could as well easily imagine a version for $X \subset \mathrm{Gr}(k, n)$ defined by zero locus of a general section of an homogeneous (ample) vector bundle.

The search for Fano varieties of K3 and CY type as complete intersections. This is probably one of the most interesting applications to me. As we recalled from the introduction, Fano varieties of K3 and CY type are important for their connections with hyperkähler geometry. On the other hand they necessarily have to be of high dimension (greater equal than four) and even higher index. This implies we have to apply theorem 8.1.4 with caution, sincer there may be some residual contributions from

the ambient space to take into account. However, there is some good news. Denote indeed by $T = \mathbb{C}[x_I, y_i]$ the basic ambient ring from which the build the Griffiths ring $\mathcal{U}$, suitably bigraded as in 8.1.1. For a variety X of dimension $2s$, a sub-structure of K3-type implies $h^{s+1,s-1}(X) = 1$ and $h^{s+t,s-t}(X) = 0$, for $t > 1$. We therefore look at $\mathcal{U}_{i,m}$, with $i \leq s-1$ having the above numerological properties. Since the relations in the Griffiths ring $\mathcal{U}$ are all in bidegree $(0,1)$ and $(1,0)$, m is negative we have for i in such a range that $T_{i,m} = \mathcal{U}_{i,m}$. This reduces the problem into a combinatorial one.
Let in fact X complete intersection of index m in the Grassmannian $\mathrm{Gr}(k, l+k)$ given by the bundle $\mathcal{F} = \bigoplus \mathcal{O}_G(d_i)$. Denote by $\alpha = c_1(\mathcal{F}) = \sum d_i$. A quick analysis of the polyonomial ring T reveals that in order to have

$$T_{s-1,m} = \mathbb{C},\ T_{s-t,m} = 0$$

the weights must be ordered as

$$d_1 > d_2 \geq \ldots \geq d_c$$

and moreover the following equation needs to be satisfied

$$2(k+l-\alpha) = d_1(kl-c-2). \tag{9.1}$$

A computer search confirms that only $X_{2,1} \subset \mathrm{Gr}(2,5)$ and $Y_1 \subset \mathrm{Gr}(3,10)$ satisfy this relation. Of course they are the well known Gushel-Mukai fourfold (see chapter 8) and the Beauville-Voisin Fano 20-fold.
However this does not rule out any other option. Thanks to the residual contributions from the Grassmannian there might be some $X_{d_1,\ldots,d_c}$ with $\mathcal{U}_{s-1,m} \neq \mathbb{C}$ but still $h^{s-1,s+1} = 1$. The condition on the ordering of the weights here might be not required. This is particularly true in the case of linear section. Indeed, after a first analysis on the cohomology groups of the ambient Grassmannian, we found a new example as

$$X_{1^4} \subset \mathrm{Gr}(2,8).$$

This is a Fano 8-fold with middle Hodge structure of K3 type. We believe it could lead to a construction of a new family of hyperkähler varieties, very likely of $\mathrm{K3}^{[n]}$ type. We compute its Hodge numbers as

Proposition 9.0.1. *Let $X_{1,1,1,1} \subset \mathrm{Gr}(2,8)$ given by a generic section of $\mathcal{O}_G(1)^{\oplus 4}$. The Hodge diamond of $X_{1,1,1,1}$ is*

$$
\begin{array}{c}
0 \quad 0 \quad 0 \quad 1 \quad 22 \quad 1 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \\
1
\end{array}
$$

with primitive $h^{4,4}_{\mathrm{prim}}(X) = 19(X)$.

Notice that the projective dual of $X_{1,1,1,1} \subset \mathrm{Gr}(2,8)$ is quartic K3 surface $S \subset \mathbb{P}^3$. However, we believe that this could be the only exception. Namely, we make the following

Conjecture 9.0.2. Let $X = X_{d_1,\dots,d_c} \subset \mathrm{Gr}(k,n)$ a Fano smooth complete intersection of even dimension (that is not a cubic fourfold). Then X is not of K3-type unless

$$(\{d_i\}, k, n) = (\{2,1\}, 2, 5), (\{1,1,1,1\}, 2, 8), (\{1\}, 3, 10).$$

We would like to extend the search to complete intersection in other homogeneous varieties as well. Moreover, we want to do a similar search for Fano varieties of CY-type. Many examples have been found in [69], but the classification is not complete at all, and our method could help in this.

Fano varieties of K3 type from homogeneous vector bundle Even if a residue theorem for homogeneous vector bundle has still to be estabilished, we started some preliminary search. In particular the equation 9.1 seems a reasonable good source of examples, when we take the bundle $\mathcal{F} = \mathcal{O}(d_1) \oplus \bigoplus \mathcal{O}(d_i) \oplus \mathbb{S}_\alpha \mathcal{S}^* \bigoplus \oplus \mathbb{S}_\beta \mathcal{Q}$. In particular all known examples in [79] satisfy this equation A lot more flexibility here is allowed. We identfied few more possible candidates. Denotes by $\iota_X = -m$ the index of X.

$\mathrm{Gr}(k,n)$	$\mathcal{F}$	dim X	ι_X	$\chi(X)$
$\mathrm{Gr}(2,9)$	$\mathcal{O}(1) \oplus \bigwedge^6 \mathcal{Q}$	6	2	24
$\mathrm{Gr}(3,8)$	$\mathcal{O}(1) \oplus \mathrm{Sym}^2 \mathcal{S}^*$	8	3	48
$\mathrm{Gr}(3,8)$	$\mathcal{O}(1) \oplus (\bigwedge^2 \mathcal{S}^*)^{\oplus 2}$	8	3	48
$\mathrm{Gr}(3,9)$	$\mathcal{O}(1) \oplus \bigwedge^2 \mathcal{S}^*$	14	6	72

All the Hodge numbers are actually computable. We have indeed the following result, whose proof can be found in the Appendix A.

Proposition 9.0.3. *Let $Z \subset \mathrm{Gr}(3,8)$ given by a generic section of $\mathcal{O}(1) \oplus \mathrm{Sym}^2 \mathcal{S}^*$. The Hodge diamond of Z is*

$$\begin{array}{ccccccccccccccccc}
0&&0&&0&&1&&24&&1&&0&&0&&0\\
&0&&0&&0&&0&&0&&0&&0&&0&\\
&&0&&0&&0&&5&&0&&0&&0&&\\
&&&0&&0&&0&&0&&0&&0&&&\\
&&&&0&&0&&3&&0&&0&&&&\\
&&&&&0&&0&&0&&0&&&&&\\
&&&&&&0&&2&&0&&&&&&\\
&&&&&&&0&&0&&&&&&&\\
&&&&&&&&1&&&&&&&&
\end{array}$$

with $h^{4,4}_{\mathrm{prim}}(Z) = 19$.

Notice that Z can be seen as a linear section of the Fano variety of plane associated to a smooth quadric hypersurface $X_2 \subset \mathbb{P}^7$. There is a natural K3 surface associated to Z. Consider in fact $Z' \subset \mathrm{OGr}(4,8)$ given by a generic section of $\bigwedge^3 \mathcal{S}^*$. Then Z' is obtained from the standard corrispondence exchanging $\bigwedge^3 \mathcal{S}^* \oplus \mathrm{Sym}^2 \mathcal{S}^* \subset \mathrm{Gr}(3,8)$ with $\bigwedge^3 \mathcal{S}^* \oplus \mathrm{Sym}^2 \mathcal{S}^* \subset \mathrm{Gr}(4,8)$. The latter is non connected, but restricting to the orthogonal Grassmannian we pick one of the two connected components. It is trivial to see that Z' defined in this way is a K3 surface.

The varieties above admits a nice interpretation as linear section of homogeneous varieties (or close to). For example the already mentioned $Z \subset \mathrm{Gr}(3,8)$ can be seen as a linear section $Z_1 \subset \mathrm{OGr}(3,8)$ of the orthogonal Grassmannian of isotropic 3-planes in a 8-dimensional space (for a fixed quadratic form), where $O(1)$ denotes the restriction of the Plücker bundle. This is an homogeneous variety that is not a Grassmannian in the classical sense (for example it has Picard rank 2). It can actually be realised as a $\mathbb{P}^3$-bundle over $\mathrm{OGr}(4,8)$.

Similarly $W \subset \mathrm{Gr}(3,9)$ defined by the zero locus of a general global section of $\mathcal{O}(1) \oplus \bigwedge^2 \mathcal{S}^*$ can be seen as a linear section $W_1 \subset \mathrm{SGr}(3,9)$, where $\mathrm{SGr}(3,9)$ denotes the odd symplectic Grassmannian of Mihai, [85].

For the other two examples, we checked with a computation similar to the one in appendix that $X \subset \mathrm{Gr}(2,9)$ given by $s \in H^0(G, \mathcal{O}(1) \oplus \bigwedge^6 \mathcal{Q})$ is of purely (p,p)-type. These results allow us to have an alternative (and quicker) computation of the Hodge numbers, since both $\mathrm{OGr}(3,8)$ and $\mathrm{SGr}(3,9)$ have cohomology purely of type (p,p), generates by Schubert classes, see for example [78]. An application of Lefschetz hyperplane section theorem and some computations of the Euler characteristics $\chi(\Omega^j_X)$ using Macaulay2 complete the computations. We collect our results in the following proposition.

Proposition 9.0.4. *1. Let $Z_1 \subset \mathrm{OGr}(3,8)$ given by a generic linear section. Then $h^{p,q} = 0$ for $p \neq q$ and $p+q < 8$. For $p+q = 8$, we have*

$$h^{5,3} = h^{3,5} = 1, \; h^{4,4} = 24,$$

with all the others middle Hodge numbers equal to zero. Moreover $h^{4,4}_{\mathrm{prim}}(Z_1) = 19$;

2. Let $W_1 \subset \mathrm{SGr}(3,9)$ given by a generic linear section. Then $h^{p,q} = 0$ for $p \neq q$ and $p+q < 14$. For $p+q = 14$, we have

$$h^{6,8} = h^{8,6} = 1, \; h^{7,7} = 26,$$

with all the others middle Hodge numbers equal to zero. Moreover $h^{7,7}_{\mathrm{prim}}(Z_1) = 20$.

The last example is still a bit mysterious. We could compute its Hodge numbers using the methods of the appendix, but this could be quite lengthy and not totally satisfying. The strategy is therefore to see $Y \subset \mathrm{Gr}(3,8)$ given by a general global section of $\mathcal{O}(1) \oplus (\bigwedge^2 \mathcal{S}^*)^{\oplus 2}$ as a linear section of some $M_\lambda \subset \mathrm{Gr}(3,8)$, where M_λ parametrise the 3-spaces in a 8 dimensional vector spaces isotropic for a pencil of skew forms. This approach has been pursued by Kuznetsov in the Lagrangian case in [82]. Some preliminary computations suggest that our Y should have $h^{4,4}(Y) = 26$, with moreover $h^{4,4}_{\mathrm{prim}}(Y) = 20$, making it a very interesting example.

We plan to investigate the geometry of these and potentially some new examples in the near future. Moreover new examples might be found, even and especially of CY-type.

Appendix A

Computations of some Hodge numbers

A.1 $Z_1 \subset \mathrm{OGr}(3,8)$ is of K3 type

We compute the Hodge numbers for $Z \subset \mathrm{Gr}(3,8)$ given by a generic

$$s \in H^0(\mathrm{Gr}(3,8), \mathrm{Sym}^2(\mathcal{S})^* \oplus \mathcal{O}(1)).$$

This is the first example coming from a computer-search based on our Griffiths-type formula, together with the condition 9.1. We verify our guess here. The calculations are a lengthy application of the Bott algorithm, together with a heavily involved diagram chasing and spectral sequences calculations. We will include many details here, in hope to serve as a reference for the similar computations. Even though in 9.0.4 we were able to compute the same number in an alternative way, we included here the computations anyway. Indeed Bott's method is far more general, and it could be applied in the future for many more cases.

Proposition A.1.1. *Let Z as above. The Hodge diamond of Z is*

$$
\begin{array}{ccccccccccccccccc}
0 & & 0 & & 0 & & 1 & & 24 & & 1 & & 0 & & 0 & & 0 \\
& 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
& & 0 & & 0 & & 0 & & 5 & & 0 & & 0 & & 0 & & \\
& & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
& & & & 0 & & 0 & & 3 & & 0 & & 0 & & & & \\
& & & & & 0 & & 0 & & 0 & & 0 & & & & & \\
& & & & & & 0 & & 2 & & 0 & & & & & & \\
& & & & & & & 0 & & 0 & & & & & & & \\
& & & & & & & & 1 & & & & & & & &
\end{array}
$$

Proof. We will divide the proof in few steps. The idea will be to compute the (simpler) Hodge diamonds of the 9-fold $Y = \mathrm{Sym}^2(\mathcal{S}^*)$, and then reduce to Z.

Computations of $\chi(\Omega^i_Z)$ and $\chi(\Omega^i_Y)$. This computations can be easily done formally using Chern classes. The package "Schubert2" of Macaulay2 is essential to speed up the calculations. One checks that

$$
\begin{aligned}
\chi(\mathcal{O}_Z) =& 1 = \chi(\mathcal{O}_Y) \\
\chi(\Omega^1_Z) =& 2 = \chi(\Omega^1_Y) \\
\chi(\Omega^2_Z) =& 3 = \chi(\Omega^2_Y) \\
\chi(\Omega^3_Z) =& 5 = \chi(\Omega^3_Y)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\chi(\Omega^4_Z) =& 24 \\
\chi(\Omega^4_Y) =& 5 \\
e_{\mathrm{top}}(Z) =& 48 \\
e_{\mathrm{top}}(Y) =& 32
\end{aligned}
$$

Let us start by computing the Hodge numbers of Y.

Computation of $h^{i,1}(Y)$. Call $\mathcal{F} = \operatorname{Sym}^2 \mathcal{S}$ the dual of the bundle we started from. Recall the conormal sequence

$$0 \to \mathcal{F}|_Y \to \Omega^1_G|_Y \to \Omega^1_Y \to 0.$$

The Koszul complex associated to s gives a resolution of $\mathcal{O}_X$:

$$0 \to \bigwedge^6 \mathcal{F} \to \bigwedge^5 \mathcal{F} \to \ldots \to \mathcal{F} \to \mathcal{O}_G \to \mathcal{O}_Y \to 0. \tag{A.1}$$

There is an associated spectral sequence as

$$\mathbf{E}_1^{-q,p} = H^p(G, \mathcal{E} \otimes \bigwedge^q \mathcal{F}) \Rightarrow H^{p-q}(G, \mathcal{E}|_Y).$$

From the conormal sequence we can pass in cohomology and deduce

$$0 \to H^1(G, \mathcal{F}|_Y) \to H^1(G, \Omega^1_G|_Y) \to H^1(Y, \Omega^1_Y) \to H^2(G, \mathcal{F}|_Y) \to H^2(G, \Omega^1_G|_Y) \to \ldots$$

We need therefore to understand

$$\mathbf{E}_1^{-q,p} = H^p(G, \mathcal{F} \otimes \bigwedge^q \mathcal{F}) \Rightarrow H^{p-q}(G, \mathcal{F}|_Y).$$

$$\mathbf{E}_1^{-q,p} = H^p(G, \Omega^1_G \otimes \bigwedge^q \mathcal{F}) \Rightarrow H^{p-q}(G, \Omega^1_G|_Y).$$

We can compute the left-hand side above using the plethysm formula of 5.2 and the Littlewood-Richardson rule 5.5 (where all the partitions in $\mathbb{S}_\lambda \mathcal{S}$ are of size at most 3, the rank or $\mathcal{S}$). First we compute

$$\begin{aligned}
\mathcal{F} =& \mathbb{S}_2\mathcal{S} \\
\bigwedge^2 \mathcal{F} =& \mathbb{S}_{3,1}\mathcal{S} \\
\bigwedge^3 \mathcal{F} =& \mathbb{S}_{3,3}\mathcal{S} \oplus \mathbb{S}_{4,1,1}\mathcal{S} \\
\bigwedge^4 \mathcal{F} =& \mathbb{S}_{4,3,1}\mathcal{S} \\
\bigwedge^5 \mathcal{F} =& \mathbb{S}_{4,4,2}\mathcal{S} \\
\bigwedge^6 \mathcal{F} =& \mathbb{S}_{4,4,4}\mathcal{S}
\end{aligned}$$

We have to compute $\Omega^1_G \otimes \bigwedge^i \mathcal{F}$ in terms of Schur functor. Recall that $\Omega^1_G \cong \mathcal{Q}^* \otimes \mathcal{S}$. We use the Littlewood-Richardson rule and deduce

$$\begin{aligned}
\Omega^1 \otimes \mathcal{F} &= \mathcal{Q}^* \otimes (\mathbb{S}_{1,1}\mathcal{S} \oplus \mathbb{S}_2\mathcal{S}) \\
\Omega^1 \otimes \bigwedge^2 \mathcal{F} &= \mathcal{Q}^* \otimes (\mathbb{S}_{3,1,1}\mathcal{S} \oplus \mathbb{S}_{3,2}\mathcal{S} \oplus \mathbb{S}_{4,1}\mathcal{S}) \\
\Omega^1 \otimes \bigwedge^3 \mathcal{F} &= \mathcal{Q}^* \otimes (\mathbb{S}_{3,3,1}\mathcal{S} \oplus \mathbb{S}_{4,3}\mathcal{S} \oplus \mathbb{S}_{4,2,1}\mathcal{S} \oplus \mathbb{S}_{5,1,1}\mathcal{S}) \\
\Omega^1 \otimes \bigwedge^4 \mathcal{F} &= \mathcal{Q}^* \otimes (\mathbb{S}_{4,3,2}\mathcal{S} \oplus \mathbb{S}_{4,4,1}\mathcal{S} \oplus \mathcal{S}_{5,3,1}\mathcal{S}) \\
\Omega^1 \otimes \bigwedge^5 \mathcal{F} &= \mathcal{Q}^* \otimes (\mathbb{S}_{4,4,3}\mathcal{S} \oplus \mathbb{S}_{5,4,2}\mathcal{S}) \\
\Omega^1 \otimes \bigwedge^6 \mathcal{F} &= \mathcal{Q}^* \otimes (\mathbb{S}_{5,4,4}\mathcal{S})
\end{aligned}$$

Thanks to Bott's algorithm it is easy to check that all the bundle above are acylic. As example of the type of calculation, consider for example $\mathcal{Q}^* \otimes \mathbb{S}_{1,1}\mathcal{S}$. This corresponds to the partition $\gamma = (0,0,0,0,-1,1,1,0)$. Let δ denotes the vector $(7,6,5,4,3,2,1,0)$. Then $\gamma + \delta = (7,6,5,4,2,3,2,0)$ contains a repeated entry and therefore the corresponding bundle is acylic. All the other computations are identical, and we will spare them to the reader. This calculation shows in turn that

$$H^i(\Omega^1_G|_Y) \cong H^i(\Omega^1_G).$$

We do a totally similar calculations for the bundles $\mathcal{F} \otimes \bigwedge^q \mathcal{F}$. They turns out to be all acyclic except for $q = 3$. Indeed the decomposition of this bundle contains the irreducible component $\mathbb{S}_{6,1,1}\mathcal{S}$. Let $\gamma = (0,0,0,0,0,6,1,1)$. The number of negative differences $\gamma_x - x < \gamma_y - y$ for $x < y$ is 5. Moreover $\mathrm{sort}(\gamma+\delta) - \delta = (1,1,1,1,1,1,1,1)$. Therefore the bundle has a unique cohomology group

$$H^5(\mathcal{F} \otimes \bigwedge^3 \mathcal{F}) \cong \mathbb{C}.$$

Plugging these informations in the conormal exact sequence we get

$$0 \to H^1(G, \Omega^1_G) \to H^1(Y, \Omega^1_Y) \to \mathbb{C} \to 0$$

and therefore

$$H^{1,1}(Y) \cong \mathbb{C}^2, \quad H^{1,i}(Y) = 0, \ i \neq 2.$$

Computation of $h^{2,i}$ and $h^{3,i}$. We now turn to the cohomolgy of the second exterior power of the cotangent bundle. The situation here is a bit more involved, if possible. Indeed the power of the conormal sequence

$$0 \to \mathrm{Sym}^2 \mathcal{F}|_Y \to \Omega^1 \otimes \mathcal{F}|_Y \to \Omega^2_G|_Y \to \Omega^2_Y \to 0$$

is not short anymore. Denote by K the kernel of the natural map $\Omega^2_G|_Y \to \Omega^2_Y \to 0$. We can split the 4-term exact sequence above in two short one, namely

$$0 \to K \to \Omega^2_G|_Y \to \Omega^2_Y \to 0; \tag{A.2}$$

$$0 \to \mathrm{Sym}^2 \mathcal{F}|_Y \to \Omega^1 \otimes \mathcal{F}|_Y \to K \to 0 \tag{A.3}$$

We have to analyse the bundles on G

$$\Omega^2_G \otimes \bigwedge^q \mathcal{F}$$
$$\Omega^1 \otimes \mathcal{F} \otimes \bigwedge^q \mathcal{F}$$
$$\mathrm{Sym}^2 \mathcal{F} \otimes \bigwedge^q \mathcal{F}$$

where we used that

$$\mathrm{Sym}^2 \mathcal{F} = \mathbb{S}_4\mathcal{S} \oplus \mathbb{S}_{2,2}\mathcal{S}$$

and

$$\Omega^2_G = (\mathbb{S}_{-2}\mathcal{Q} \otimes \mathbb{S}_{1,1}\mathcal{S}) \oplus (\mathbb{S}_{-1,-1}\mathcal{Q} \otimes \mathbb{S}_2\mathcal{S}).$$

The bundle $\Omega^2_G \otimes \bigwedge^q \mathcal{F}$ is acyclic for $q \geq 1, q \neq 4$. For $q = 4$ there is the irreducible components $\mathbb{S}_{-1,-1}\mathcal{Q} \otimes \mathbb{S}_{6,3,1}\mathcal{S}$. This has a unique cohomology group

$$H^7(\Omega^2_G \otimes \bigwedge^4 \mathcal{F}) \cong \mathbb{C}.$$

The bundle $\Omega^1 \otimes \mathcal{F} \otimes \bigwedge^q \mathcal{F}$ is acylic for $q \neq 3$. For $q = 3$ the bundle contains two copies of $\mathcal{Q}^* \otimes \mathbb{S}_{6,2,1}\mathcal{S}$, with unique cohomology group

$$H^6(\Omega^1 \otimes \mathcal{F} \otimes \bigwedge^3 \mathcal{F}) \cong \mathbb{C}^2.$$

The bundle $\mathrm{Sym}^2(\mathcal{F}) \otimes \bigwedge^q \mathcal{F}$ has cohomology for $(p, q) = (5, 1), (5, 2), (5, 3)$. We have in particular

$$\begin{aligned} H^5(\mathrm{Sym}^2 \mathcal{F} \otimes \mathcal{F}) &\cong \mathbb{C}^{28} \\ H^5(\mathrm{Sym}^2(\mathcal{F}) \otimes \bigwedge^2 \mathcal{F}) &\cong \mathbb{C}^{65} \\ H^5(\mathrm{Sym}^2(\mathcal{F}) \otimes \bigwedge^3 \mathcal{F}) &\cong \mathbb{C}^{36} \end{aligned}$$

The only non trivial spectral sequence is therefore for the last bundle. We can compute the cohomology of $\mathrm{Sym}^2 \mathcal{F}|_Y$ using the above spectral sequence or as follows. Consider the Koszul complex A.1 twisted by $\mathrm{Sym}^2 \mathcal{F}$. Split it in six short exact sequences

$$\begin{aligned} &0 \to L_1 \to \mathrm{Sym}^2 \mathcal{F} \to \mathrm{Sym}^2 \mathcal{F}|_Y \to 0 \\ &0 \to L_2 \to \mathrm{Sym}^2 \mathcal{F} \otimes \mathcal{F} \to L_1 \to 0 \\ &\vdots \\ &0 \to \mathrm{Sym}^2 \mathcal{F} \otimes \bigwedge^6 \mathcal{F} \to \mathrm{Sym}^2 \mathcal{F} \otimes \bigwedge^6 \mathcal{F} \to L_5 \to 0 \end{aligned}$$

Using the above cohomology computations, one gets $h^4(\mathrm{Sym}^2 \mathcal{F}|_Y) - h^3(\mathrm{Sym}^2 \mathcal{F}|_Y) = (h^5(L_2) - 28) - (h^5(L_2) + 36 - 64) = 0$. The long exact sequence in cohomology associated to A.3 therefore gives $h^2(K) - h^3(K) = -2$. This implies in A.2 that

$$h^{2,2}(Y) - h^{2,3}(Y) = 3, \text{ and } h^{2,i}(Y) = 0, \; i \neq 2, 3.$$

Of course we would like to prove that $h^{2,3} = 0$. To do this we have to repeat the calculations for $h^i\Omega^3$. The calculations are indentical in the phylosophy to the previous case, therefore we will skip most of the details. We have first to split the sequence

$$0 \to \operatorname{Sym}^3 \mathcal{F}|_Y \to \Omega^1 \otimes \operatorname{Sym}^2 \mathcal{F}|_Y \to \Omega^2 \otimes \mathcal{F}|_Y \to \Omega^3|_Y \to \Omega_Y \to 0$$

into short exact ones

$$\begin{aligned}
&0 \to \operatorname{Sym}^3 \mathcal{F}|_Y \to \Omega^1 \otimes \operatorname{Sym}^2 \mathcal{F}|_Y \to K_2 \to 0\\
&0 \to K_2 \to \Omega^2 \otimes \mathcal{F}|_Y \to K_1 \to 0\\
&0 \to K_1 \to \Omega^3|_Y \to \Omega_Y \to 0
\end{aligned}$$

Then we determine the decomposition of the various bundle in terms of Schur powers, namely

$$\begin{aligned}
\operatorname{Sym}^3 \mathcal{S} &= \mathbb{S}_6\mathcal{S} \oplus \mathbb{S}_{4,2}\mathcal{S} \oplus \mathbb{S}_{2,2,2}\mathcal{S}\\
\Omega^3_G =&(\mathbb{S}_{-3}\mathcal{Q} \otimes \mathbb{S}_{1,1,1}\mathcal{S}) \oplus (\mathbb{S}_{-1,-2}\mathcal{Q} \otimes \mathbb{S}_{2,1}\mathcal{S}) \oplus (\mathbb{S}_{-1,-1,-1}\mathcal{Q} \otimes \mathbb{S}_3\mathcal{S})
\end{aligned}$$

Suppose that a bundle $\mathcal{Q} \otimes \bigwedge^q \mathcal{F}$ has a unique cohomology group in degree p, with dimension k. Call (p, q, k) an *admissible triple*. The set of the admissible triples for the bundles we consider are

$\mathcal{Q} \otimes \bigwedge^q \mathcal{F}$	(p, q, k)
Ω^3_G	$(4, 3, 2)$
$\Omega^2_G \otimes \mathcal{F}$	$(5, 1, 70), (6, 2, 28), (7, 3, 65), (7, 4, 36), (8, 5, 5)$
$\operatorname{Sym}^2 \mathcal{F} \otimes \Omega^1_G$	$(6, 3, 72), (6, 2, 130), (5, 1, 420)$
$\operatorname{Sym}^3 \mathcal{F}$	$(5, 3, 330), (5, 2, 960), (5, 1, 1008), (5, 0, 28)$

One computes that $h^4(\operatorname{Sym}^2 \mathcal{F} \otimes \Omega^1|_Y) - h^3(\operatorname{Sym}^2 \mathcal{F} \otimes \Omega^1|_Y) = 362$, $h^4(\Omega^2 \otimes \mathcal{F}|_Y) = 13$, $h^3(\operatorname{Sym}^3 \mathcal{F}|_Y) - h^4(\operatorname{Sym}^3 \mathcal{F}|_Y) = 378$. From the sequences above one gets $-h^{3,3}(Y) + h^{4,3}(Y) + 7 = h^4(K_1) - h^3(K_1)$. The difference on the right hand side equals

$$-h^4(\operatorname{Sym}^2 \mathcal{F} \otimes \Omega^1|_Y) + h^3(\operatorname{Sym}^2 \mathcal{F} \otimes \Omega^1|_Y) - h^4(\Omega^2 \otimes \mathcal{F}|_Y) + h^3(\operatorname{Sym}^3 \mathcal{F}|_Y) - h^4(\operatorname{Sym}^3 \mathcal{F}|_Y) - 1 = 2.$$

Since we already know that $\chi(\Omega^3_Y) = -5$ we have

$$-h^{3,3} + h^{4,3}(Y) = -5, h^{i,3}(Y) = 0, i \neq 3, 4.$$

By Lefschetz hyperplane theorem $h^{p,q}(Z) = h^{p,q}(Y)$ for $p + q < 8$. Since $\chi(\Omega^3_Z) = -6$ and $e_{\text{top}}(Z) = 48$ and finally this implies

$$h^{3,5}(Z) = 1, h^{4,4}(Z) = 24, h^{3,4} = 0.$$

□

Corollary A.1.2. *From the above proof it follows that the only non-zero Hodge numbers for Y are*

$$h^{0,0} = 1, h^{1,1} = 2, h^{2,2} = 3, h^{3,3} = 5, h^{4,4} = 5, h^{5,5} = 5, h^{6,6} = 5, h^{7,7} = 3, h^{8,8} = 2, h^{9,9} = 1.$$

Bibliography

[1] H. Abe and T. Matsumura. Equivariant cohomology of weighted Grassmannians and weighted Schubert classes. *Int. Math. Res. Not. IMRN*, (9):2499–2524, 2015.

[2] S. Altınok. *Graded rings corresponding to polarised K3 surfaces and Fano 3 folds.* PhD thesis, University of Warwick, 1998.

[3] S. Altınok. Constructing new $K3$ surfaces. *Turkish J. Math.*, 29(2):175–192, 2005.

[4] S. Altınok, G. Brown, and M. Reid. Fano 3-folds, $K3$ surfaces and graded rings. In *Topology and geometry: commemorating SISTAG*, volume 314 of *Contemp. Math.*, pages 25–53. Amer. Math. Soc., Providence, RI, 2002.

[5] M. F. Atiyah. Complex analytic connections in fibre bundles. *Trans. Amer. Math. Soc.*, 85:181–207, 1957.

[6] L. S. Badescu. *Projective geometry and formal geometry*, volume 65. Birkhäuser, 2012.

[7] S. A. Barannikov. *Extended moduli spaces and mirror symmetry in dimensions $n >$* 3. ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–University of California, Berkeley.

[8] R. Barlow. A simply connected surface of general type with $p_g = 0$. *Inventiones mathematicae*, 79(2):293–301, 1985.

[9] V. V. Batyrev, D. A. Cox, et al. On the hodge structure of projective hypersurfaces in toric varieties. *Duke Mathematical Journal*, 75(2):293–338, 1994.

[10] I. C. Bauer, F. Catanese, and R. Pignatelli. Complex surfaces of general type: some recent progress. In *Global aspects of complex geometry*, pages 1–58. Springer, 2006.

[11] I. C. Bauer, F. Catanese, and R. Pignatelli. The moduli space of surfaces with $K^2 = 6$ and $p_g = 4$. *Math. Ann.*, 336(2):421–438, 2006.

[12] K. Behrend and B. Noohi. Moduli of non-commutative polarized schemes. *ArXiv e-prints*, July 2015.

[13] V. Benedetti. Manifolds of low dimension with trivial canonical bundle in grassmannians. *arXiv preprint arXiv:1609.02695*, 2016.

[14] Y. Bespalov, V. Lyubashenko, and O. Manzyuk. *Pretriangulated A_∞-categories.* Pratsi Instytutu Matematyky Natsional'noï Akademiï Nauk Ukraïny. Matematyka ta ïï Zastosuvannya 76. Kyïv: Instytut Matematyky NAN Ukraïny. 598 p. , 2008.

[15] I. Biswas and G. Schumacher. On the stability of the tangent bundle of a hypersurface in a Fano variety. *J. Math. Kyoto Univ.*, 45(4):851–860, 2005.

[16] R. Blache. Chern classes and Hirzebruch-Riemann-Roch theorem for coherent sheaves on complex-projective orbifolds with isolated singularities. *Math. Z.*, 222(1):7–57, 1996.

[17] J. Boehm and A. Frübis-Krüger. A smoothness test for higher codimensions. preprint available online as arXiv:1602.04522, 2016.

[18] L. Borisov and A. Căldăraru. The pfaffian-grassmannian derived equivalence. *Journal of Algebraic Geometry*, 18(2):201–222, 2009.

[19] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[20] R. Bott. Homogeneous vector bundles. *Annals of Mathematics*, pages 203–248, 1957.

[21] G. E. Bredon. *Sheaf theory*, volume 170. Springer Science & Business Media, 2012.

[22] G. Brown and E. Fatighenti. Hodge numbers and deformations of Fano 3-folds. *ArXiv e-prints*, July 2017.

[23] G. Brown and K. Georgiadis. Polarized calabi–yau 3-folds in codimension 4. *Mathematische Nachrichten*, 290(5-6):710–725, 2017.

[24] G. Brown and A. M. Kasprzyk. The graded ring database. online. access via http://www.grdb.co.uk/.

[25] G. Brown and A. M. Kasprzyk. Orbifold projective hypersurfaces in dimension ≤ 4. 2014.

[26] G. Brown, M. Kerber, and M. Reid. Fano 3-folds in codimension 4, Tom and Jerry. Part I. *Compos. Math.*, 148(4):1171–1194, 2012.

[27] G. Brown, M. Kerber, and M. Reid. Tom and Jerry: Big Table, 2012. Linked at `http://grdb.co.uk/Downloads/`.

[28] G. Brown and K. Suzuki. Computing certain fano 3-folds. *Japan journal of industrial and applied mathematics*, 24(3):241–250, 2007.

[29] G. Brown and K. Suzuki. Fano 3-folds with divisible anticanonical class. *Manuscripta Math.*, 123(1):37–51, 2007.

[30] R. L. Bryant, S.-S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior differential systems*, volume 18. Springer Science & Business Media, 2013.

[31] R.-O. Buchweitz and H. Flenner. The global decomposition theorem for Hochschild (co-)homology of singular spaces via the Atiyah-Chern character. *Adv. Math.*, 217(1):243–281, 2008.

[32] A. Buckley, M. Reid, and S. Zhou. Ice cream and orbifold Riemann-Roch. *Izv. Ross. Akad. Nauk Ser. Mat.*, 77(3):29–54, 2013.

[33] A. Calabri, C. Ciliberto, and M. Mendes Lopes. Numerical godeaux surfaces with an involution. *Transactions of the American Mathematical Society*, 359(4):1605–1632, 2007.

[34] A. Căldăraru. The Mukai pairing. II. The Hochschild-Kostant-Rosenberg isomorphism. *Adv. Math.*, 194(1):34–66, 2005.

[35] J. A. Carlson and P. A. Griffiths. Infinitesimal variations of hodge structure and the global torelli problem. journées de géometrie algbrique d'angers, juillet 1979/algebraic geometry. In *51–76, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md.* Citeseer, 1979.

[36] I. Cheltsov and J. Park. Birationally rigid fano threefold hypersurfaces. *arXiv preprint arXiv:1309.0903*, 2013.

[37] J.-J. Chen, J. A. Chen, and M. Chen. On quasismooth weighted complete intersections. *J. Algebraic Geom.*, 20(2):239–262, 2011.

[38] J. A. Christophersen and N. O. Ilten. Vanishing cotangent cohomology for plücker algebras. *Communications in Algebra*, pages 1–19, 2017.

[39] C. H. Clemens. Double solids. *Adv. in Math.*, 47(2):107–230, 1983.

[40] A. Corti, A. Pukhlikov, and M. Reid. Fano 3-fold hypersurfaces. In *Explicit birational geometry of 3-folds*, volume 281 of *London Math. Soc. Lecture Note Ser.*, pages 175–258. Cambridge Univ. Press, Cambridge, 2000.

[41] A. Corti and M. Reid, editors. *Explicit birational geometry of 3-folds*, volume 281 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2000.

[42] A. Corti and M. Reid. Weighted Grassmannians. In *Algebraic geometry*, pages 141–163. de Gruyter, Berlin, 2002.

[43] S. Coughlan, L. Golebiowski, G. Kapustka, and M. Kapustka. Arithmetically gorenstein calabi-yau threefolds in $\mathbb{P}^7$. *arXiv preprint arXiv:1609.01195*, 2016.

[44] D. A. Cox and M. L. Green. Polynomial structures and generic torelli for projective hypersurfaces. *Compositio Mathematica*, 73(2):121–124, 1990.

[45] D. A. Cox and S. Katz. *Mirror symmetry and algebraic geometry*, volume 68. American Mathematical Society Providence, RI, 1999.

[46] P. De Poi and F. Zucconi. On subcanonical Gorenstein varieties and apolarity. *J. Lond. Math. Soc. (2)*, 87(3):819–836, 2013.

[47] O. Debarre. On prime Fano varieties of degree 10 and coindex 3, 2012.

[48] O. Debarre, A. Iliev, and L. Manivel. On the period map for prime Fano threefolds of degree 10. *J. Algebraic Geom.*, 21(1):21–59, 2012.

[49] O. Debarre and A. Kuznetsov. Gushel–mukai varieties: classification and birationalities. *arXiv preprint arXiv:1510.05448*, 2015.

[50] O. Debarre and C. Voisin. Hyper-kähler fourfolds and grassmann geometry. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2010(649):63–87, 2010.

[51] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann. Singular 4-0-2 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de, 2015.

[52] C. Di Natale, E. Fatighenti, and D. Fiorenza. Hodge theory and deformations of affine cones of subcanonical projective varieties. preprint available online as arXiv:1512.00835, 2015.

[53] A. Dimca et al. *Residues and cohomology of complete intersections.* Sonderforschungsbereich Geometrie und Analysis, 1994.

[54] I. Dolgachev. Weighted projective varieties. In *Group actions and vector fields (Vancouver, B.C., 1981)*, volume 956 of *Lecture Notes in Math.*, pages 34–71. Springer, Berlin, 1982.

[55] R. Donagi. Generic torelli for projective hypersurfaces. *Compositio Mathematica*, 50(2-3):325–353, 1983.

[56] R. Donagi et al. On the geometry of grassmannians. *Duke Math. J*, 44(4):795–837, 1977.

[57] R. Donagi and L. W. Tu. Generic torelli for weighted hypersurfaces. *Mathematische Annalen*, 276(3):399–413, 1987.

[58] D. Eisenbud and D. Buchsbaum. Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3. *Amer. J. Math.*, 99:447–485, 1977.

[59] E. Fatighenti, L. Rizzi, and F. Zucconi. Weighted fano varieties and infinitesimal torelli problem. available at arXiv:1611.05355, 2016.

[60] M. Filip. Hochschild cohomology and deformation quantization of affine toric varieties. *arXiv preprint arXiv:1706.00580*, 2017.

[61] H. Flenner. Divisorenklassengruppen quasihomogener Singularitäten. *J. Reine Angew. Math.*, 328:128–160, 1981.

[62] D. R. Grayson and M. E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[63] M. L. Green. The period map for hypersurface sections of high degree of an arbitrary variety. *Compositio Mathematica*, 55(2):135–156, 1985.

[64] P. Griffiths. Periods of integrals on algebraic manifolds, i, ii, iii. *American Journal of Mathematics*, 90(3):805–865, 1968.

[65] A. Grothendieck. *Cohomologie locale des faisceaux cohérents... (SGA 2)*, volume 2 of *Advanced Studies in Pure Mathematics*. North-Holland Publishing Co., 1968.

[66] D. Halpern-Leistner. Lefschetz Hyperplane Theorem for stacks. preprint available online as arXiv:1008.0891, 2010.

[67] F. Hirzebruch. *Topological Methods in Algebraic Geometry*. Springer-Verlag Berlin Heidelberg, 1978.

[68] A. R. Iano-Fletcher. Working with weighted complete intersections. In *Explicit birational geometry of 3-folds*, volume 281 of *London Math. Soc. Lecture Note Ser.*, pages 101–173. Cambridge Univ. Press, Cambridge, 2000.

[69] A. Iliev and L. Manivel. Fano manifolds of calabi–yau hodge type. *Journal of Pure and Applied Algebra*, 219(6):2225–2244, 2015.

[70] N. O. Ilten. Versal deformations and local Hilbert schemes. *J Softw. Algebra Geom.*, (4):12–16, 2012.

[71] D. Inoue, A. Ito, and M. Miura. Complete intersection calabi–yau manifolds with respect to homogeneous vector bundles on grassmannians. *arXiv preprint arXiv:1607.07821*, 2016.

[72] A. Kanazawa. Pfaffian calabi-yau threefolds and mirror symmetry. *Commun. Number Theory Phys*, 6(3):661–696, 2012.

[73] Y. Kawamata. Boundedness of $\mathbf{Q}$-Fano threefolds. In *Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989)*, volume 131 of *Contemp. Math.*, pages 439–445. Amer. Math. Soc., Providence, RI, 1992.

[74] S. Kebekus, T. Peternell, A. J. Sommese, and J. A. Wiśniewski. Projective contact manifolds. *Inventiones mathematicae*, 142(1):1–15, 2000.

[75] I.-K. Kim, T. Okada, and J. Won. Alpha invariants of birationally rigid fano threefolds. *arXiv preprint arXiv:1604.00252*, 2016.

[76] J. Kollár, Y. Miyaoka, S. Mori, and H. Takagi. Boundedness of canonical $\mathbf{Q}$-Fano 3-folds. *Proc. Japan Acad. Ser. A Math. Sci.*, 76(5):73–77, 2000.

[77] K. Konno. On the variational torelli problem for complete intersections. *Compositio Mathematica*, 78(3):271–296, 1991.

[78] A. Kresch and H. Tamvakis. Quantum cohomology of orthogonal grassmannians. *Compositio Mathematica*, 140(2):482–500, 2004.

[79] O. Küchle. On fano 4-folds of index 1 and homogeneous vector bundles over grassmannians. *Mathematische Zeitschrift*, 218(1):563–575, 1995.

[80] A. Kuznetsov. Küchle fivefolds of type c5. *Mathematische Zeitschrift*, 284(3-4):1245–1278, 2016.

[81] A. Kuznetsov and D. Markushevich. Symplectic structures on moduli spaces of sheaves via the atiyah class. *Journal of Geometry and Physics*, 59(7):843–860, 2009.

[82] A. G. Kuznetsov. On Küchle varieties with picard number greater than 1. *Izvestiya: Mathematics*, 79(4):698, 2015.

[83] L. Manivel. On the period map for prime fano threefolds of degree 10. *Journal of Algebraic Geometry*, 21(1):21–59, 2012.

[84] L. Manivel. On fano manifolds of picard number one. *Mathematische Zeitschrift*, 281(3-4):1129–1135, 2015.

[85] I. A. Mihai. Odd symplectic flag manifolds. *Transformation groups*, 12(3):573–599, 2007.

[86] S. Mori. On 3-dimensional terminal singularities. *Nagoya Math. J.*, 98:43–66, 1985.

[87] S. Mori. Flip theorem and the existence of minimal models for 3-folds. *J. Amer. Math. Soc.*, 1(1):117–253, 1988.

[88] S. Mukai. Curves and Grassmannians. In *Algebraic geometry and related topics (Inchon, 1992)*, Conf. Proc. Lecture Notes Algebraic Geom., I, pages 19–40. Int. Press, Cambridge, MA, 1993.

[89] T. Okada. Stable rationality of orbifold fano threefold hypersurfaces. *arXiv preprint arXiv:1608.01186*, 2016.

[90] S. A. Papadakis. The equations of type II_1 unprojection. *J. Pure Appl. Algebra*, 212(10):2194–2208, 2008.

[91] T. Peternell and J. A. Wisniewski. On stability of tangent bundles of fano manifolds with b_2= 1. *arXiv preprint alg-geom/9306010*, 1993.

[92] M. Pizzato, T. Sano, and L. Tasin. Effective non-vanishing for fano weighted complete intersections. *arXiv preprint arXiv:1703.07344*, 2017.

[93] V. L. Popov and E. B. Vinberg. Invariant theory. In *Algebraic geometry IV*, pages 123–278. Springer, 1994.

[94] Y. Prokhorov and M. Reid. On q-fano threefolds of fano index 2. *arXiv preprint arXiv:1203.0852*, 2012.

[95] G. V. Ravindra and V. Srinivas. The Grothendieck-Lefschetz theorem for normal projective varieties. *J. Algebraic Geom.*, 15(3):563–590, 2006.

[96] M. Reid. Canonical 3-folds. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 273–310. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[97] M. Reid. Minimal models of canonical 3-folds. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 131–180. North-Holland, Amsterdam, 1983.

[98] M. Reid. The moduli space of 3-folds with $K = 0$ may nevertheless be irreducible. *Math. Ann.*, 278(1-4):329–334, 1987.

[99] M. Reid. Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.

[100] M. Reid. Campedelli versus godeaux. *Problems in the theory of surfaces and their classification (Cortona, 1988)*, 309:365, 1991.

[101] M. Reid. Graded rings and birational geometry, 2001.

[102] M. Reid. Fun in codimension 4. preprint available online via the author's webpage, 2011.

[103] E. A. Rødland. The pfaffian calabi–yau, its mirror, and their link to the grassmannian g (2, 7). *Compositio Mathematica*, 122(2):135–149, 2000.

[104] M.-H. Saito. Generic torelli theorem for hypersurfaces in compact irreducible hermitian symmetric spaces. *Algebraic geometry and commutative algebra: in honor of Masayoshi Nagata*, 2:615, 1988.

[105] M.-H. Saito et al. Weak global torelli theorem for certain weighted projective hypersurfaces. *Duke Math. J*, 53(1):67–111, 1986.

[106] T. Sano. On deformations of $\mathbb{Q}$-Fano 3-folds. *J. Algebraic Geom.*, 25(1):141–176, 2016.

[107] M. Schlessinger. Functors of artin rings. *Transactions of the American Mathematical Society*, 130:208–222, 1968.

[108] M. Schlessinger. On rigid singularities. *Rice Univ. Studies*, 59(1):147–162, 1973. Complex analysis, 1972 (Proc. Conf., Rice Univ., Houston, Tex., 1972), Vol. I: Geometry of singularities.

[109] E. Sernesi. *Deformations of algebraic schemes*, volume 334. Springer Science & Business Media, 2007.

[110] D. M. Snow. Cohomology of twisted holomorphic forms on grassmann manifolds and quadric hypersurfaces. *Mathematische Annalen*, 276(1):159–176, 1986.

[111] J. H. M. Steenbrink. Mixed Hodge structure on the vanishing cohomology. In *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, pages 525–563. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[112] J. Stevens. *Deformations of singularities*, volume 1811 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.

[113] H. Takagi. On classification of $\mathbb{Q}$-fano 3-folds of gorenstein index 2. i. *Nagoya Mathematical Journal*, 167:117–155, 2002.

[114] F. Tonoli. Construction of Calabi-Yau 3-folds in $\mathbb{P}^6$. *J. Algebraic Geom.*, 13(2):209–232, 2004.

[115] L. Tu. Macaulay's theorem and local torelli for weighted hypersurfaces. *Compositio Mathematica*, 60(1):33–44, 1986.

[116] C. Voisin. *Hodge Theory and Complex Algebraic Geometry, II.* Cambridge, Cambridge, 2003.

[117] J. Wahl. The Jacobian algebra of a graded Gorenstein singularity. *Duke Mathematical Journal*, 55(4):843–871, 1987.

[118] J. Weyman. *Cohomology of vector bundles and syzygies*, volume 149. Cambridge University Press, 2003.

[119] F. Xu. *On the smooth linear section of the Grassmannian G(2,n).* PhD thesis, Rice Univeristy.